Configurations of Extremal Even Unimodular Lattices

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Key Terms

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  - free $\mathbb{Z}$-module

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  - free $\mathbb{Z}$-module
  - with an inner product $\langle \cdot , \cdot \rangle : L \times L \to \mathbb{R}$

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  - rank

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- **Lattice** L — “integer vector space”
  - rank
    - number of basis vectors

Key Terms

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  - even

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  - even
    - $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$

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    - $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$
    - all vectors have even squared length

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    - L is "self-dual"
    - L's matrix has determinant 1

Key Terms

- Review: **even unimodular lattice** L

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- Review: **even unimodular lattice** L
  - all vectors have even squared length

Key Terms

- Review: **even unimodular lattice** L
  - "self-dual"

Key Terms

- Review: **even unimodular** **lattice** L
  - "integer vector space"

Key Terms

- **even unimodular lattice** L...

Key Terms

- **even unimodular lattice** $L\ldots$
  - “rare”

Key Terms

- **even unimodular lattice** L...
  - “rare”
    - $\mathrm{rank}(L) = 8n$

Key Terms

- **even unimodular lattice** L...

Key Terms

- **even unimodular lattice** L…
  - extremal

Key Terms

- **even unimodular lattice** L...
  - extremal
    - shortest vector

Key Terms

- **even unimodular lattice** L...
  - extremal
    - shortest vector $\longleftrightarrow$ as long as possible

We care about...

- *extremal* **even unimodular lattices**

We care because...

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    - Chemical lattices (in dimensions $n \leq 3$)
    - (Continuous) communication decoding

Methods: Theta Functions

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- How:

  “dot product $\langle x, x\rangle$” $\iff$ “length”

- What:

$$\Theta_L(\tau) = \sum_{x \in L} e^{i\pi\tau\langle x,x\rangle} = \sum_{k=1}^{\infty} a_k e^{i\pi\tau(k)}$$

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- Example:

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$$\Theta_{\mathbb{Z}^2}(\tau) = \sum_{x\in L} e^{i\pi\tau\langle x,x\rangle} = 1 + 4e^{i\pi\tau} + 4e^{2i\pi\tau} + 4e^{4i\pi\tau} + \cdots$$

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  ...even if we cannot write down a basis for L.

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- What we do:

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  ...which give combinatorial information about lattice vectors.

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Theorem.

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Theorem. For L even unimodular and extremal of rank $n \in \{8, 24, 32, 48, 56, 72, 96, 120\}$, the minimal-norm vectors of L generate L.

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Such lattices are *not* known to exist for $n \in \{72, 96, 120\}$.

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Theorem. For L even unimodular and extremal of rank $n \in \{8, 24, 32, 48, 56, 72, 96, 120\}$, the minimal-norm vectors of L generate L.

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Bad news: The analogous result fails for $n = 40$.

Good news: It is *almost* true.

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Theorem. Let $m(L)$ be the minimal norm of vectors in an extremal even unimodular lattice L of rank n. The vectors in L having norms $m(L)$ and $(m(L)+2)$ generate L.

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Note: This bound is sharp for $n \in \{16, 40, 80\}$

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Answer: In principle, yes.

Problem: Computationally intensive....

stluseR

Theorem. If L is the unique rank-72 even unimodular lattice with automorphisms given by $SL_2(\mathbb{Z}/71\mathbb{Z})$ then

$$\Theta_L = 1 + 71712q^3 + 6213012336q^4 + 15281966487168q^5 + \cdots .$$

stluseR

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(Just proven by **N. D. Elkies, Z. Abel, & S. D. Kominers.**)

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- Configuration results for non-extremal even unimodular lattices of ranks $n \in \{56, 72, 96, 120\}$?

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- Theta series of the unique rank-80 even unimodular lattice with automorphisms given by $SL_2(\mathbb{Z}/79\mathbb{Z})$?

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- Configuration results for non-extremal even unimodular lattices of ranks $n \in \{56, 72, 96, 120\}$?
- Theta series of the unique rank-80 even unimodular lattice with automorphisms given by $SL_2(\mathbb{Z}/79\mathbb{Z})$?
- Rank-72 extremal even unimodular lattices?

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Questions?