Respect for Improvements and Comparative Statics in Matching Markets*

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Abstract

One of the oldest results in the theory of two-sided matching is the *entry comparative static*, which shows that under the Gale–Shapley deferred acceptance algorithm, adding a new agent to one side of the market makes all the agents on the other side weakly better off. Here, we give a new proof of the entry comparative static, by way of a well-known property of deferred acceptance called *respect for improvements*. Our argument extends to yield comparative static results in more general settings, such as the matching with slot-specific preferences framework.

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1 Introduction

The theory of stable matching—first introduced by Gale and Shapley (1962) in the *American Mathematical Monthly*—has by now been applied in a wide range of settings including medical labor markets (Roth (1984a); Roth and Peranson (1999)) and public school choice (Balinski and Sönmez (1999); Abdulkadiroğlu and Sönmez (2003); Pathak (2011)).[1] The question Gale and Shapley (1962) asked is simple: given a set of men and women, each with preferences over members of the opposite gender, is it possible to find a *stable matching*, i.e., an assignment of partners such that no one finds his or her partner unacceptable, and no two agents mutually prefer each other to their assigned partners?

Gale and Shapley (1962) answered their question in the affirmative by providing a constructive algorithm for finding stable matchings, *deferred acceptance* (which we review in Section 2.2). Subsequent authors characterized the set of stable matchings (McVitie and Wilson (1971); Knuth (1976); Roth and Sotomayor (1990); Roth (1984a, 1986); Balinski and Ratier (1998)) and examined the incentives agents face under deferred acceptance (Dubins and Freedman (1981); Roth (1982); Demange et al. (1987)).[2] All of the aforementioned results have been generalized to far more complex settings: Gale and Shapley (1962) themselves showed how to generalize one-to-one "marriage" matching to a many-to-one model of "college admissions" matching, in which each student attends at most one college but colleges may accept multiple students. Roth (1984b), Blair (1988), Sotomayor (1999), Baıou and Balinski (2000) and Echenique and Oviedo (2006) extended the model further to many-to-many matching settings (e.g., matching workers with firms). Crawford and Knoer (1981), Kelso and Crawford (1982), Fleiner (2003), Hatfield and Milgrom (2005), and Hatfield and Kominers (2017a) considered two-sided markets in which the matching process determines not just

[1] Stable matching was also the subject of the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel (also known as the Nobel Memorial Prize in Economic Sciences; see Royal Swedish Academy of Sciences (2012)).

[2] Deferred acceptance is "strategy-proof" for one side of the market, in the sense that it incentivizes agents on that side of the market to reveal their true preferences; this is in some sense best-possible, as Roth (1982) showed that no stable matching mechanism can be strategy-proof for both sides of the market at the same time (see also Sönmez (1997)).

who matches with whom, but also *contracts* over terms of exchange such as wages or hours worked; Ostrovsky (2008) extended this work to multi-layered supply chain settings (see also Hatfield and Kominers (2012)). Hatfield et al. (2013, 2018a, forthcoming) and Fleiner et al. (2018, forthcoming) have generalized still further to trading networks.

One of the most classic results in matching theory is the *entry comparative static*, which compares stable matchings before and after the entry of new agents to the market (Kelso and Crawford (1982); Gale and Sotomayor (1985); Roth and Sotomayor (1990); Crawford (1991)). The exact form of the entry comparative static varies across settings (see, e.g., Blum et al. (1997); Blum and Rothblum (2002); Hatfield and Milgrom (2005); Biró et al. (2008); Ostrovsky (2008); Hatfield and Kominers (2013); Chambers and Yenmez (2017); Yenmez (2018); Fleiner et al. (2018)), but the core result is that under deferred acceptance, adding a new agent to one side of the market makes all agents on the other side of the market weakly better off.[3]

Proofs of the entry comparative static and its generalizations tend to rely either on meticulous inductive arguments (Kelso and Crawford (1982); Crawford (1991); Blum et al. (1997); Blum and Rothblum (2002); Hatfield and Milgrom (2005); Biró et al. (2008); Hatfield and Kominers (2013); Fleiner et al. (2018)) or on broad results on the structure of the set of stable matchings (Gale and Sotomayor (1985); Ostrovsky (2008); Chambers and Yenmez (2017)). Here, we give a new, more concise proof by way of another classic result—specifically, the fact that deferred acceptance *respects improvements* in the sense that making one agent more highly ranked in other agents' preferences improves that agent's deferred acceptance outcome (Balinski and Sönmez (1999)). As we describe, our approach extends directly to yield comparative static results in more general settings, such as the "slot-specific preferences" framework Kominers and Sönmez (2016) introduced to model matching with affirmative action/diversity constraints.

[3]As we discuss in Section 5, we typically also obtain a sort of dual to this result: agents on the same side of the market as the entering agent are made worse off. To our knowledge, however, the methods we use here only enable us to derive the comparative static for agents on the side of the market opposite to that of the entering agent.

2 The Marriage Model

We start by introducing the marriage model of Gale and Shapley (1962): There are finite sets M and W of *men* and *women*; we denote by $I \equiv M \cup W$ the set of *agents*. We assume that each man $m \in M$ is has a complete, transitive, and strict *preference ordering* $\succ_m$ over $W \cup \{\emptyset\}$, where $\emptyset$ denotes an *outside option* that represents the possibility of remaining unmatched. Similarly, each woman $w \in W$ has a complete, transitive, and strict *preference ordering* $\succ_w$ over $M \cup \{\emptyset\}$. For each agent $i \in I$, we denote the weak part of i's preferences by $\succcurlyeq_i$, so that if $j \succcurlyeq_i k$ then either $j \succ_i k$ or $j = k$. We use the convention that $\succ_{I'} \equiv (\succ_i)_{i \in I'}$ and $\succcurlyeq_{I'} \equiv (\succcurlyeq_i)_{i \in I'}$.

2.1 Stable Matchings

A *matching* is a map $\mu : I \to I \cup \{\emptyset\}$ from the set of agents to the set of agents plus the outside option, such that:

1. Under μ, each man is assigned to a woman or to the outside option—that is, $\mu(m) \in (W \cup \{\emptyset\})$ for each $m \in M$—and each woman is assigned to a man or to the outside option—$\mu(w) \in (M \cup \{\emptyset\})$ for each $w \in W$.

2. If a man $m \in M$ is assigned to a woman $w \in W$, then w is assigned to m—i.e., if $\mu(m) = w \in W$, then $\mu(w) = M$—and vice versa—if $\mu(w) = m \in M$, then $\mu(m) = w$.

We say that agent $i \in I$ is *matched to* $j \in I$ *under* μ if $\mu(i) = j$; we say that $i \in I$ is *unmatched under* μ if $\mu(i) = \emptyset$.

A matching μ is *individually rational* if no agent prefers the outside option to his or her assigned partner, i.e., if for all $i \in I$, we have that $\mu(i) \succcurlyeq_i \emptyset$. A matching μ is *unblocked* if there does not exist a man $m \in M$ and woman $w \in W$ who mutually prefer each other to their assigned match partners, i.e., such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. A matching μ is *stable* if it is both individually rational and unblocked.

2.2 Deferred Acceptance and the Existence of Stable Matchings

Gale and Shapley (1962) gave a constructive proof that stable outcomes exist, via the following *(man-proposing) deferred acceptance algorithm*:

> **Step** 1. Each man proposes to his first-choice woman. Each woman *holds* her best acceptable proposal (if any) and rejects all other proposals.
>
> **Step** $\ell \geq 2$. Each man who was rejected in the previous step proposes to his most-preferred woman to whom he has not yet proposed (if any). Each woman holds her best acceptable proposal among those made in this round and held from the previous round (if any) and rejects all others.
>
> If at any time no men are available to make proposals—that is, if all men not currently held have proposed to all women they find acceptable—then the algorithm terminates. The *(man-proposing) deferred acceptance outcome* is the matching that matches each woman to the man whose proposal she is holding (if any) at the end of the last step before the algorithm terminates.[4]

The deferred acceptance outcome is always stable: We have individual rationality because men only propose to women they find acceptable, and women only hold acceptable proposals. Meanwhile, if there were a blocking pair $(m, w) \in M \times W$, then it would have to the case that m proposed to w over the course of the algorithm (as he prefers w to his deferred acceptance partner); this means that w must have rejected m at some step, as otherwise w would be m's partner under the deferred acceptance outcome. But then, w must have received a proposal superior to m's over the course of the algorithm—so we cannot have that w prefers m to her deferred acceptance partner, contradicting the possibility that (m, w) is a blocking pair.

[4]The term "deferred acceptance" derives from the fact that the acceptance step is deferred until the end.

3 The Classic Comparative Static

Now, we consider what happens to the deferred acceptance outcome when we add a new woman $\tilde{w}$ to the market. Abusing notation slightly, we extend our model to the set of agents $I \cup \{\tilde{w}\} = M \cup W \cup \{\tilde{w}\}$, writing $\tilde{\succ}$ and $\tilde{\mu}$, respectively, for preferences and matchings in the expanded market. To reflect the idea that $\tilde{w}$ is a new entrant to an existing market, we require that the preference profile $\tilde{\succ}$ be consistent with $\succ$ on I:

$$\text{for all } i, j, k \in I, \text{ we have } j \mathrel{\tilde{\succ}_i} k \iff j \succ_i k;$$

equivalently, for each man $m \in M$, the expanded preferences $\tilde{\succ}_m$ correspond to $\succ_m$ with $\tilde{w}$ inserted somewhere, and $\tilde{\succ}_W = \succ_W$.

A classic comparative static result shows that the entry of woman $\tilde{w}$ must make all the men weakly better off under man-proposing deferred acceptance.

Theorem 1 (Kelso and Crawford (1982); Gale and Sotomayor (1985); Roth and Sotomayor (1990, p. 44))**.** *If μ^* is the outcome of man-proposing deferred acceptance in the market I and $\tilde{\mu}^*$ is the outcome of man-proposing deferred acceptance in the market $I \cup \{\tilde{w}\}$ that arises after the the entry of woman $\tilde{w}$, then each man $m \in M$ (weakly) prefers his assignment under $\tilde{\mu}^*$ to his assignment under μ^*; that is,*

$$\tilde{\mu}^*(m) \mathrel{\tilde{\succeq}_m} \mu^*(m).^5 \tag{1}$$

Here, we give a novel proof of Theorem 1 based on a property of deferred acceptance called respect for (unambiguous) improvements.

[5] Theorem 1 also arises as a special case of the comparative static results of Crawford (1991), Hatfield and Milgrom (2005), Ostrovsky (2008), Hatfield and Kominers (2013), Chambers and Yenmez (2017), Yenmez (2018), and Fleiner et al. (2018).

3.1 Respect for Improvements

For $w \in W$ and preference relations $\hat{\succ}_w$ and $\check{\succ}_w$, we say that $\hat{\succ}_w$ *unambiguously improves upon* $\check{\succ}_w$ *for* $m \in M$ if m is ranked (weakly) higher under $\hat{\succ}_w$ than under $\check{\succ}_w$, and the relative rankings of all other men are left unchanged. Formally: $\hat{\succ}_w$ is an *unambiguous improvement over* $\check{\succ}_w$ *for* $m \in M$ if

- for all $m' \in ((M \setminus \{m\}) \cup \{\emptyset\})$, if $m \check{\succ}_w m'$ then $m \hat{\succ}_w m'$, and
- for all $m', m'' \in (M \setminus \{m\})$, we have $m' \check{\succ}_w m''$ if and only if $m' \hat{\succ}_w m''$.

We say that a profile of the women's preferences $\hat{\succ}_W$ is an *unambiguous improvement over* $\check{\succ}_W$ *for* m if each $\hat{\succ}_w$ is an unambiguous improvement over $\check{\succ}_w$ for m.

Balinski and Sönmez (1999) showed that deferred acceptance *respects unambiguous improvements* in the sense that if $\hat{\succ}_W$ is an unambiguous improvement over $\check{\succ}_W$ for m, then m (weakly) prefers his man-proposing deferred acceptance outcome when the women's preferences are $\hat{\succ}_W$ to that when the women's preferences are $\check{\succ}_W$.

Lemma 1 (Balinski and Sönmez (1999)). *If* $\hat{\succ}_W$ *is an unambiguous improvement over* $\check{\succ}_W$ *for* m*, and* $\check{\mu}^*$ *and* $\hat{\mu}^*$ *denote the man-proposing deferred acceptance outcomes under* $(\succ_M, \hat{\succ}_W)$ *and* $(\succ_M, \check{\succ}_W)$*, respectively, then* m *(weakly) prefers his or her assignment under* $\hat{\mu}^*$ *to that under* $\check{\mu}^*$*; that is,*

$$\hat{\mu}^*(m) \succcurlyeq_m \check{\mu}^*(m).$$

Respect for unambiguous improvements is an extremely natural property to desire in practice, as it means that agents have no incentive to try to lower their standing in others' preference relations.[6,7] But the respect for improvements condition is also subtle: because the definition of "unambiguous improvement" does not allow $m' = \emptyset$, a (strict) unambiguous improvement for one man can also be a (strict) unambiguous improvement for some other

[6] Imagine the chaos if self-sabotage were a standard feature of the marriage market!

[7] Moreover, respect for unambiguous improvements in some sense uniquely characterizes deferred acceptance (see Theorems 5 and 6 of Balinski and Sönmez (1999)).

man, so long as both men's ranks just improve relative to the outside option $\emptyset$. This property allows us to derive an elegant and quick proof of our comparative static result, Theorem 1.

3.2 Proof of Theorem 1

As in the theorem statement, we suppose that μ^* is the outcome of man-proposing deferred acceptance in the market I and $\tilde{\mu}^*$ is the outcome of man-proposing deferred acceptance in the market $I \cup \{\tilde{w}\}$.

We write the preferences of $\tilde{w}$ in the form

$$\tilde{\succ}_{\tilde{w}} : m_1 \,\tilde{\succ}_{\tilde{w}} \cdots \tilde{\succ}_{\tilde{w}} \, m_\ell \,\tilde{\succ}_{\tilde{w}} \, \emptyset \,\tilde{\succ}_{\tilde{w}} \, m_{\ell+1} \cdots , \tag{2}$$

and let $\bar{\succ}_{\tilde{w}}$ be the "null" preference relation for $\tilde{w}$ that is consistent with $\tilde{\succ}_{\tilde{w}}$ but treats all men as unacceptable:

$$\bar{\succ}_{\tilde{w}} : \emptyset \,\bar{\succ}_{\tilde{w}} \, m_1 \,\bar{\succ}_{\tilde{w}} \cdots \bar{\succ}_{\tilde{w}} \, m_\ell \,\bar{\succ}_{\tilde{w}} \, m_{\ell+1} \cdots . \tag{3}$$

We let $\bar{\mu}^*$ be the outcome of man-proposing deferred acceptance in the market with agents $I \cup \{\tilde{w}\}$ and preference profile $(\tilde{\succ}_M, \tilde{\succ}_W, \bar{\succ}_{\tilde{w}})$. As $\tilde{\succ}_W = \succ_W$ and each $\tilde{\succ}_m$ agrees with $\succ_m$ except for the inclusion of $\tilde{w}$, we see directly that

$$\bar{\mu}^*(i) = \mu^*(i) \tag{4}$$

for each $i \in I$. Indeed, as $\tilde{w}$ treats all men as unacceptable under $\bar{\succ}_{\tilde{w}}$, deferred acceptance's outcome is unchanged if all men drop $\tilde{w}$ from their preference relations—and with that preference adjustment, the algorithm corresponds exactly to deferred acceptance in the market I, and so must yield an outcome that corresponds to μ^*.

Now, we note by inspecting (2) and (3) that $\tilde{\succ}_{\tilde{w}}$ is an unambiguous improvement over $\bar{\succ}_{\tilde{w}}$ for each man $m \in M$. Hence, by Lemma 1 we see that each man $m \in M$ (weakly) prefers his assignment under $\tilde{\mu}^*$ to that under $\bar{\mu}^*$; that is,

$$\tilde{\mu}^*(m) \tilde{\succsim}_m \bar{\mu}^*(m) \tag{5}$$

for each man $m \in M$. Combining (4) and (5) shows the desired result, (1).

3.3 Discussion

Many proofs of Theorem 1 (and its generalizations) proceed by starting with the result of man-proposing deferred acceptance in the original market and then studying how the market re-equilibrates after $\tilde{w}$ enters (see, e.g., Kelso and Crawford (1982); Blum et al. (1997); Blum and Rothblum (2002); Hatfield and Milgrom (2005); Biró et al. (2008); Hatfield and Kominers (2013); Fleiner et al. (2018)).[8] Another approach explicitly tracks how the presence of $\tilde{w}$ affects each stage of deferred acceptance (Crawford (1991); Chambers and Yenmez (2017)). Our argument avoids the need to explicitly track the path of adjustment. At the same time, our approach highlights that the key feature underlying Theorem 1 is that $\tilde{w}$'s entry simultaneously improves all the men's overall standing in the market by increasing competition among the women.

Meanwhile, our proof of Theorem 1 is closely related to a much earlier argument due to Gale and Sotomayor (1985). Like us, Gale and Sotomayor (1985) treated $\tilde{w}$'s entry as equivalent to transforming $\tilde{w}$'s preferences from a null relation to $\tilde{\succ}_{\tilde{w}}$. Unlike in our argument, however, Gale and Sotomayor (1985) derived the comparative static through appeal to the lattice structure of stable outcomes (see Roth and Sotomayor (1990, pp. 43–44)). Ostrovsky (2008) likewise relied on lattice structure in proving a version of Theorem 1 for supply chains. Approaching the argument in terms of respect for improvements is useful because it immediately suggests substantial generalizations, some of which are outside the scope of the lattice structure result—as we describe next.

[8]Dworczak (2018) tracks similar re-equilibration dynamics in his *Deferred Acceptance with Compensation Chains* algorithms, which generalize deferred acceptance by allowing agents on both sides to make offers.

4 A More General Model: Matching with Slot-Specific Preferences

We now extend Theorem 1 to a setting with *many-to-one matching*—that is, one in which employees on one side of the market may take multiple partners on the other side (Gale and Shapley (1962); Roth (1985); Roth and Sotomayor (1990)). We furthermore generalize by allowing the matching process to determine not just who matches with whom but also *contracts* that specify terms of exchange like wages or hours worked (Crawford and Knoer (1981); Kelso and Crawford (1982); Hatfield and Milgrom (2005)). Specifically, we work with the *slot-specific preference* structure introduced by Kominers and Sönmez (2016).

4.1 Intuition

The slot-specific preferences framework, which we describe formally in the next section, is a model of employee–firm matching in which each firm has a set of positions—*slots*—that can be assigned to different employees. Slots have their own (potentially independent) rankings over contracts. Within each firm, a linear order called the *order of precedence* determines the order in which slots are filled.

One special case of slot-specific preferences is the *responsive preference* model of Roth (1985), in which firms' preferences are consistent with a single linear order over contracts and a maximum number of positions that can be filled.[9] More broadly, slot-specific preferences embed multiple models of affirmative action, such as using some slots to reserve positions for members of disadvantaged groups (Abdulkadiroğlu (2005); Kojima (2012); Hafalir et al. (2013); Kominers and Sönmez (2014); Dur, Pathak and Sönmez (2016); Dur et al. (2018)).

[9] The responsive preference model arises by making all the slots within a given firm identical.

4.2 Formal Model

We suppose that there is a set of *employees* E and a set of *firms* F, and a (finite) set of *contracts* X. Each contract $x \in X$ is between an employee $\mathsf{e}(x) \in E$ and firm $\mathsf{f}(x) \in F$, and may also specify additional "terms" of exchange drawn from a set T. Thus X may be considered a subset of $E \times F \times T$. We extend the notations $\mathsf{e}(\cdot)$ and $\mathsf{f}(\cdot)$ to sets of contracts $Y \subseteq X$ by setting $\mathsf{e}(Y) \equiv \cup_{y \in Y}\{\mathsf{e}(y)\}$ and $\mathsf{f}(Y) \equiv \cup_{y \in Y}\{\mathsf{f}(y)\}$. For $Y \subseteq X$, we denote $Y_e \equiv \{y \in Y : \mathsf{e}(y) = e\}$ and $Y_{E'} \equiv \cup_{e \in E'} Y_e$; analogously, we denote $Y_f \equiv \{y \in Y : \mathsf{f}(y) = f\}$ and $Y_{F'} \equiv \cup_{f \in F'} Y_f$.

Each employee $e \in E$ has a complete, transitive, and strict preference order $\succ_e$ (with weak order $\succcurlyeq_e$) over contracts in $X_e \cup \{\emptyset\}$, where $X_e \equiv \{x \in X : \mathsf{e}(x) = e\}$ and, as before, $\emptyset$ is an "outside option," which represents remaining unmatched; we use the convention that $\emptyset \succ_e x$ for all $x \in X \setminus X_e$. We say that the contracts $x \in X$ for which $\emptyset \succ_e x$ are *unacceptable to* e. We denote the profile of all employees' preferences by $\succ_E$.

Each firm $f \in F$ has a set S_f of *slots*; each slot can be assigned at most one contract in $X_f \equiv \{x \in X : \mathsf{f}(x) = f\}$. Slots $s \in S_f$ have (linear) preference orders $\succ_s$ (with weak orders $\succcurlyeq_s$) over contracts in X_f. For convenience, we use the convention that $Y_s \equiv Y_f$ for $s \in S_f$. As with employees, we assume that each slot s ranks an outside option $\emptyset$ which represents remaining unassigned, and as with employees, we use the convention that $\emptyset \succ_s x$ if $x \in X \setminus X_s$. We set $S \equiv \cup_{f \in F} S_f$ and denote the profile of all slots' preferences by $\succ_S$.

Employees have *unit demand*, that is, they choose at most one contract from a set of contract offers. We assume also that employees always choose the best available contract, so that the choice $C^e(Y)$ of an employee $e \in E$ from contract set $Y \subseteq X$ is the $\succ_e$-maximal element of Y_e (or the outside option if $\emptyset \succ_e y$ for all $y \in Y_e$).[10]

Meanwhile, firms $f \in F$ may be assigned as many as $q_f \equiv |S_f|$ contracts—one for each slot in S_f—but may hold no more than one contract with a given employee. We assume that

[10] To simplify our exposition and notation, we treat individual contracts as interchangeable with singleton contract sets.

for each $f \in F$, the slots in S_f are ordered according to a (linear) *order of precedence* $\rhd^f$. We denote $S_f \equiv \left\{s_f^1, \ldots, s_f^{q_f}\right\}$ with the understanding that $s_f^\ell \rhd^f s_f^{\ell+1}$ unless otherwise noted. The interpretation of $\rhd^f$ is that if $s \rhd^f s'$ then—whenever possible—firm f fills slot s before filling s'. Formally, the choice $C^f(Y)$ of a firm $f \in F$ from contract set $Y \subseteq X$ is defined as follows:

- First, slot s_f^1 is assigned the contract x^1 that is $\succ_{s_f^1}$-maximal among contracts in Y.
- Then, slot s_f^2 is assigned the contract x^2 that is $\succ_{s_f^2}$-maximal among contracts in the set $Y \setminus Y_{\mathsf{e}(x^1)}$ of contracts in Y with employees other than $\mathsf{e}(x^1)$.
- This process continues in sequence, with each slot s_f^ℓ being assigned the contract x^ℓ that is $\succ_{s_f^\ell}$-maximal among contracts in the set $Y \setminus Y_{\mathsf{e}(\{x^1,\ldots,x^{\ell-1}\})}$.

If no contract $x \in Y$ is assigned to slot $s_f^\ell \in S_f$ in the computation of $C^f(Y)$, then s_f^ℓ is assigned the null contract $\emptyset$.

4.3 Stable Outcomes

An *outcome* is a set of contracts $Y \subseteq X$ that is "feasible" in the sense that

- Y contains at most one contract for each employee, i.e., $|Y_e| \leq 1$ for each $e \in E$, and
- Y contains at most q_f contracts for each firm f, i.e., $|Y_e| \leq q_f$ for each $f \in F$.

We say that an outcome Y is *stable* if it is

1. *individually rational*—$C^e(Y) = Y_e$ for all $e \in E$ and $C^f(Y) = Y_f$ for all $f \in F$—and
2. *unblocked*—there does not exist a firm $f \in F$ and *blocking set* $Z \neq C^f(Y)$ such that $Z = C^f(Y \cup Z)$ and $Z_e = C^e(Y \cup Z)$ for all $e \in \mathsf{e}(Z)$.

Note that if all employee–firm pairs may contract and the set T of contractual terms is trivial, then $X = E \times F$. If moreover all workers and firms have unit demand (i.e., if we always have $|C^e(Y)| \leq 1$ and $|C^f(Y)| \leq 1$), then we recover the marriage model from Section 2, where the man–woman pairs matched under some matching μ correspond to the pairs contained in an outcome Y.

4.4 Generalized Deferred Acceptance

Kominers and Sönmez (2016) showed that stable outcomes exist under slot-specific preferences, and can be found via the following *(employee-proposing) cumulative offer process* (Kelso and Crawford (1982); Hatfield and Milgrom (2005)) that generalizes deferred acceptance:[11]

Step 1**.** Some employee $e^1 \in E$ proposes his or her most-preferred contract, $x^1 \in X_{e^1}$. Firm $\mathsf{f}(x^1)$ *holds* x^1 if $x^1 \in C^{\mathsf{f}(x^1)}(\{x^1\})$, and *rejects* x^1 otherwise. Set $A^2_{\mathsf{f}(x^1)} = \{x^1\}$, and set $A^2_{f'} = \varnothing$ for each $f' \neq \mathsf{f}(x^1)$; these are the sets of contracts *available* to firms at the beginning of Step 2.

Step $\ell \geq 2$**.** Some employee $e^\ell \in E$ for whom no contract is currently held by any firm proposes his or her most-preferred contract that has not yet been rejected, $x^\ell \in X_{e^\ell}$. Firm $\mathsf{f}(x^\ell)$ holds the contracts in $C^{\mathsf{f}(x^\ell)}(A^\ell_{\mathsf{f}(x^\ell)} \cup \{x^\ell\})$ and rejects all other contracts in $A^\ell_{\mathsf{f}(x^\ell)} \cup \{x^\ell\}$; firms $f' \neq \mathsf{f}(x^\ell)$ continue to hold all contracts they held at the end of Step $\ell - 1$. Set $A^{\ell+1}_{\mathsf{f}(x^\ell)} = A^\ell_{\mathsf{f}(x^\ell)} \cup \{x^\ell\}$, and set $A^{\ell+1}_{f'} = A^\ell_{f'}$ for each $f' \neq \mathsf{f}(x^\ell)$.

If at any time no employee is able to propose a new contract—that is, if all employees for whom no contracts are on hold have proposed all contracts they find acceptable—then the algorithm terminates. The *outcome of the (employee-proposing) cumulative offer process* is the set of contracts held by firms at the end of the last step before the algorithm terminates.

[11] The cumulative offer process generalizes deferred acceptance by (in principle) allowing firms to hold contracts they had rejected in earlier steps—although under slot-specific preferences, firms never actually use this extra degree of freedom (see Kominers and Sönmez (2016); Hatfield and Kominers (2017b); Hatfield et al. (2018b)). For consistency with Kominers and Sönmez (2016), we state the cumulative offer process with a single agent proposing in each step; Hirata and Kasuya (2014) showed that in our setting this formulation is equivalent to one in which employees propose simultaneously, directly generalizing the version of deferred acceptance we presented in Section 2.2.

Although stable outcomes exist under slot-specific preferences, the set of stable outcomes under slot-specific preferences does not have lattice structure (see Kominers and Sönmez (2016) and also Hatfield and Kominers (2017b)). Consequently the approach Gale and Sotomayor (1985) used to prove their comparative static result does not carry over to the slot-specific preference setting.[12] Nevertheless, as we show in the sequel, our approach based on respect for improvements extends directly.

4.5 Respect for Improvements Under Slot-Specific Preferences

Following Kominers and Sönmez (2016), we say that preference profile $\hat{\succ}_S$ is an *unambiguous improvement over* $\check{\succ}_S$ *for* $e \in E$ if for all slots $s \in S$:

1. for all $x \in X_e$ and $y \in (X_{E \setminus \{e\}} \cup \{\emptyset\})$, if $x \check{\succ}_s y$, then $x \hat{\succ}_s y$; and

2. for all $y, z \in X_{E \setminus \{e\}}$, $y \hat{\succ}_s z$ if and only if $y \check{\succ}_s z$.

That is, $\hat{\succ}_S$ is an unambiguous improvement over preference profile $\check{\succ}_S$ for $e \in E$ if $\hat{\succ}_S$ is obtained from $\succ$ by raising the positions of some of e's contracts (at some slots) while leaving the relative preference orders of other employees' contracts unchanged.

The cumulative offer process *respects unambiguous improvements* in the sense that if $\hat{\succ}_S$ is an unambiguous improvement over $\check{\succ}_S$ for e, then e (weakly) prefers his or her cumulative offer process outcome under $(\succ_E, \hat{\succ}_S)$ to that under $(\succ_E, \check{\succ}_S)$.

Lemma 2 (Kominers and Sönmez (2016))**.** *If $\hat{\succ}_S$ is an unambiguous improvement over $\check{\succ}_S$ for e, and $\check{Y}^*$ and $\hat{Y}^*$ denote the cumulative offer process outcomes under $(\succ_E, \hat{\succ}_S)$ and $(\succ_E, \check{\succ}_S)$, respectively, then e (weakly) prefers his or her assignment under $\hat{Y}^*$ to that under $\check{Y}^*$; that is,*

$$\hat{Y}^*_e \succcurlyeq_e \check{Y}^*_e.$$

[12]That said, it is possible that we could indirectly apply the Gale and Sotomayor (1985) approach by passing through an associated one-to-one matching market that Kominers and Sönmez (2016) construct in order to analyze the cumulative offer mechanism under slot-specific preferences.

4.6 Comparative Statics

4.6.1 Expanding Capacity

Lemma 2 immediately implies comparative statics for markets with slot-specific preferences—generalizing Theorem 1—through a version of the argument we presented in Section 3.2. Here, the result is at the level of adding slots to the market.

We consider what happens to the cumulative offer process outcome when we add a slot $\tilde{s}$ at firm f. Abusing notation again, we extend our slot-specific preference model to the set of slots $\tilde{S} = S \cup \{\tilde{s}\}$, writing $\tilde{\succ}_S$ and $\tilde{Y}$, respectively, for slot preferences and outcomes in the expanded market ($\tilde{s}$ can appear anywhere in the precedence order). To reflect the idea that $\tilde{s}$ is a new addition to an existing market, we require that preferences of slots in S are unchanged: $\tilde{\succ}_S = \succ_S$. (Note that here we may also leave the employee preferences $\succ_E$ unchanged, since we have not in any way changed the set of firms F or the set of contracts X.)

By construction, adding the new slot $\tilde{s}$ impacts the market exactly the way that adding a new woman impacted the market in Section 3.2: it is as if we raised the position of all agents' contracts at a slot that previously treated all contracts as unacceptable, holding all other slots' preferences fixed. Hence, adding $\tilde{s}$ results in an unambiguous improvement for all agents; this implies the following comparative static.

Theorem 2. *If Y^* is the outcome of the employee-proposing cumulative offer process in the market with the set of slots S and $\tilde{Y}^*$ is the outcome of the employee-proposing cumulative offer process in the market with the set of slots $\tilde{S} = S \cup \{\tilde{s}\}$, then each employee $e \in E$ (weakly) prefers his or her assignment under $\tilde{Y}^*$ to his or her assignment under Y^*; that is,*

$$\tilde{Y}^*_e \succeq_e Y^*_e.$$

Theorem 2 generalizes Theorem 1, as entry of a new woman into a marriage market can be modeled in the slot-specific preferences framework as adding a single slot to a firm that previously had none. Theorem 2 also has practical implications: Perhaps most naturally, the result means that under deferred acceptance, expanding the number of available positions at a firm always works in employees' favor. Moreover, by applying Theorem 2 iteratively (adding one slot at a time), we obtain the conclusion of Theorem 2 for the entry of a wholly new employer to the market.

4.6.2 Adding Contracts

A further generalization of Theorem 2 shows that adding new contracts at the bottom of some slots' preference orders (that is, right before the null contracts $\emptyset$) again results in improved outcomes for all employees.

Suppose that we introduce new contracts $\tilde{X}$, yielding the contract set $X \cup \tilde{X}$. We write $\tilde{\succ}$ and $\tilde{Y}$ for preferences and outcomes in the expanded market, assuming that $y \mathrel{\tilde{\succ}_r} y'$ if and only if $y \succ_r y'$ for all $r \in E \cup S$ and $y, y' \in X$—reflecting the idea that adding $\tilde{x}$ does not affect agents' or slots' preferences over pre-existing contract options. If $y \mathrel{\tilde{\succ}_s} \tilde{x}$ for all $s \in S$, $y \in X$, and $\tilde{x} \in \tilde{X}$, then, just as in the argument we used in Section 3.2, we may interpret $\tilde{\succ}_S$ as an unambiguous improvement over $\succ_S$ under the contract set $X \cup \tilde{X}$ by imagining that $\succ_S$ ranks all the contracts in $\tilde{X}$ as unacceptable—with relative ranking consistent with $\tilde{\succ}_S$—so that $\tilde{\succ}_S$ can be obtained from $\succ_S$ by raising the position of contracts in $\tilde{X}$ relative to the outside option. Thus, we have the following comparative static by Lemma 2.

Theorem 3. *If Y^* is the outcome of the employee-proposing cumulative offer process in the market with the set of contracts X and $\tilde{Y}^*$ is the outcome of the employee-proposing cumulative offer process in the market with the set of contracts $X \cup \tilde{X}$ (with $y \mathrel{\tilde{\succ}_s} \tilde{x}$ for all $s \in S$, $y \in X$, and $\tilde{x} \in \tilde{X}$), then each employee e (weakly) prefers his or her assignment under $\tilde{Y}^*$ to his or her assignment under Y^*; that is,*

$$\tilde{Y}^*_e \mathrel{\tilde{\succeq}_e} Y^*_e.$$

Lastly, we note that adding new contracts for a single agent e *anywhere in slots' preferences* also results in an unambiguous improvement—and hence ensures that e will be better off under the cumulative offer process. (We cannot in general add contracts at arbitrary positions for multiple agents, however, as the resulting change in preferences might not be an unambiguous improvement.)

Indeed, suppose that we introduce a new contract $\tilde{x}$, yielding the contract set $X \cup \{\tilde{x}\}$. We write $\tilde{\succ}$ and $\tilde{Y}$ for preferences and outcomes in the expanded market, assuming that $y \mathrel{\tilde{\succ}_r} y'$ if and only if $y \succ_r y'$ for all $r \in E \cup S$ and $y, y' \in X$—again reflecting the idea that adding $\tilde{x}$ does not affect agents' or slots' preferences over pre-existing contract options. By construction, adding the new contract $\tilde{x}$ results in an unambiguous improvement for $\mathsf{e}(\tilde{x})$: employee $\mathsf{e}(\tilde{x})$ has weakly-higher ranked contracts at every slot, while relative rankings of all other contracts are unchanged; this implies the following comparative static by Lemma 2.

Theorem 4. *If Y^* is the outcome of the employee-proposing cumulative offer process in the market with the set of contracts X and $\tilde{Y}^*$ is the outcome of the employee-proposing cumulative offer process in the market with the set of contracts $X \cup \{\tilde{x}\}$, then $\mathsf{e}(\tilde{x})$ (weakly) prefers his or her assignment under $\tilde{Y}^*$ to his or her assignment under Y^*; that is,*

$$\tilde{Y}^*_{\mathsf{e}(\tilde{x})} \mathrel{\tilde{\succsim}_{\mathsf{e}(\tilde{x})}} Y^*_{\mathsf{e}(\tilde{x})}.$$

4.7 Discussion

Chambers and Yenmez (2017) showed a result analogous to Theorem 2 for ways of "expanding" choice rules that satisfy a regularity condition called *path-independence* (Aizerman and Malishevski (1981)). Yenmez (2018) extended the Chambers and Yenmez (2017) result still further to cover choice rules that can be modified—or in the language of Hatfield and Kominers (2017b), *completed*—in ways that make them path-independent (see also Echenique and Yenmez (2015); Kamada and Kojima (2015)). Slot-specific preferences are not path-independent, in general, but *do* have path-independent completions (see Hatfield and Kominers (2017b)). However, the additions of slots and contracts considered in Theorems 2–4 are *not* expansions in the sense of Chambers and Yenmez (2017) and Yenmez (2018); hence, our results here extend Chambers and Yenmez's (2017) and Yenmez's (2018) comparative statics to new types of transformations. To our knowledge, we are the first to prove comparative statics for adding new contracts to the market.

Our use of the respect for improvements result (Lemma 2) here is similar to an early application Kominers and Sönmez (2014) presented for the slot-specific preferences framework. Indeed, Kominers and Sönmez (2014) use respect for improvements under slot-specific preferences to show that guaranteeing slots at a school for minorities (Hafalir et al. (2013)) improves welfare relative to simply capping the number of majority students allowed to attend that school (Kojima (2012)). The crux of the Kominers and Sönmez (2014) argument consists of observing that converting a quota slot into a reserve slot is an unambiguous improvement for all majority students, as it corresponds to raising majority students' position relative to the null option at each quota slot (while still ranking majority students below minority students at those slots); this is analogous to the way we apply Lemma 2.

5 Conclusion

Using the *respect for improvements* property of deferred acceptance, we have developed a new method of proving entry comparative statics in matching markets. Our method generalizes readily to any matching setting for which a respect for improvements result is known: We illustrated one such generalization in Section 4, where we used respect for improvements to show comparative statics for matching settings with slot-specific preferences. Likewise, we can extend the entry comparative static to matching settings under weakened substitutability conditions using the respect for improvements results of Afacan (2017) and Ma and Kominers (2019).

We note, however, that others have often found more refined comparative static results through appeal to structural results for stable matchings. Most common is a sort of dual to Theorem 1, showing that entry of a new woman to the market makes all the other women (weakly) worse off (see, e.g., Gale and Sotomayor (1985); Roth and Sotomayor (1990); Crawford (1991)). Others have given more precise characterizations of which agents are helped and/or harmed by other agents' entry (see, e.g., Romm (2014)), and/or shown comparative statics for the full set of stable matchings, rather than just for the deferred acceptance outcome (see, e.g., Blum et al. (1997); Chambers and Yenmez (2017)). It is unclear whether our respect for improvements-based approach can also be used to prove the refinements just described. At minimum, such an argument would seem to require a sharper version of the respect for improvements theorem; we leave this for future work.

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