Stable and Strategy-Proof Matching with Flexible Allotments

By John William Hatfield, Scott Duke Kominers, and Alexander Westkamp*

Firms with multiple divisions often face global budget constraints that make each division's ability to hire contingent on the number of workers hired by other divisions. For example, hospitals must consider how hiring one type of specialist affects the resources available for hiring other types. Similarly, collegiate sports teams often face hard caps on the number of athletic scholarships they can offer, so offering a scholarship to a player at a given position leaves one less scholarship for other positions. And universities often face trade-offs in hiring across different academic disciplines.

In this paper, we show how to model firms with cross-division constraints using the framework of matching with contracts (Kelso and Crawford, 1982; Hatfield and Milgrom, 2005). In addition to covering the applications above, our framework nests the baseline Kamada and Kojima (2015, 2017) model of matching with distributional constraints and the Kominers and Sönmez (2016) model of matching with slot-specific priorities.

Stable and strategy-proof mechanisms have proven vital for practical matching applications (see, e.g., Roth (2008), Pathak and Sönmez (2008), and Hassidim et al. (2017)). But cross-division constraints introduce new complexities that render prior approaches to proving stability and strategy-proofness inapplicable (including the approaches of Hatfield and Milgrom (2005), Hatfield and Kojima (2010), Kominers and Sönmez (2016), and Hatfield and Kominers (2016)). Nevertheless, we are able to show that stable and strategy-proof matching is possible in the presence of cross-division constraints; to do this, we build upon our recent work (Hatfield et al., 2016) on *observable substitutability.*

I. Matching with Contracts

We start with the standard model of *many-to-one matching with contracts*: There is a finite set of *doctors* D and a finite set of *hospitals* H. There is a finite set of *contracts* X; each $x \in X$ is associated with a doctor $\mathsf{d}(x)$ and a hospital $\mathsf{h}(x)$. There may be many contracts for each doctor–hospital pair. We call a set of contracts $Y \subseteq X$ an *outcome*, with $\mathsf{d}(Y) \equiv \cup_{y \in Y}\{\mathsf{d}(y)\}$ and $\mathsf{h}(Y) \equiv \cup_{y \in Y}\{\mathsf{h}(y)\}$. For any $i \in D \cup H$, we let $Y_i \equiv \{y \in Y : i \in \{\mathsf{d}(y), \mathsf{h}(y)\}\}$. We say that an outcome $Y \subseteq X$ is *feasible* if $|Y_d| \leq 1$ for all $d \in D$.

Each doctor $d \in D$ has unit demand over contracts in X_d and an *outside option* $\emptyset$. We denote the strict preferences of doctor d over $X_d \cup \{\emptyset\}$ by $\succ_d$. A contract $x \in X_d$ is *acceptable for d (with respect to* $\succ_d$*)* if $x \succ_d \emptyset$. We extend doctor preferences over contracts to preferences over outcomes in the natural way.

Each hospital $h \in H$, meanwhile, has multi-unit demand, and is endowed with a choice function C^h that describes how h would choose from any offered set of contracts. We assume throughout that for all $Y \subseteq X$ and all $h \in H$, the choice function C^h

(1) only selects contracts to which h is a party, i.e., $C^h(Y) \subseteq Y_h$,

(2) selects at most one contract with any given doctor, i.e., $C^h(Y)$ is feasible, and

(3) satisfies the *irrelevance of rejected contracts* condition of Aygün and Sönmez (2013),

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i.e., for all $z \in X$, if $z \notin C^h(Y \cup \{z\})$, then $C^h(Y \cup \{z\}) = C^h(Y)$.

For any $Y \subseteq X$ and $h \in H$, we denote by $R^h(Y) \equiv Y \smallsetminus C^h(Y)$ the set of contracts that h *rejects* from Y.

A. *Stability*

A feasible outcome $A \subseteq X$ is *stable* if it is

1) *Individually rational*: $C^h(A) = A_h$ for all $h \in H$, and $A \succeq_d \emptyset$ for all $d \in D$.
2) *Unblocked*: There does not exist a nonempty $Z \subseteq (X \smallsetminus A)$ such that $Z_h \subseteq C^h(A \cup Z)$ for all $h \in \mathsf{h}(Z)$, and $Z \succ_d A$ for all $d \in \mathsf{d}(Z)$.

Our definition of stability requires that no agent wishes to abrogate one or more contracts unilaterally, and that there does not exist a *blocking set* Z such that all hospitals and doctors associated with contracts in Z prefer to sign the contracts in Z (potentially after dropping some of their contracts in A).

B. *Mechanisms*

A *mechanism* $\mathcal{M}(\cdot; C)$ maps preference profiles $\succ = (\succ_d)_{d \in D}$ to outcomes, given a profile of hospital choice functions $C = (C^h)_{h \in H}$. Throughout, we assume that the choice functions of the hospitals are fixed and write $\mathcal{M}(\succ)$ in place of $\mathcal{M}(\succ; C)$.

A mechanism $\mathcal{M}$ is *stable* if $\mathcal{M}(\succ)$ is a stable outcome for every preference profile $\succ$. A mechanism $\mathcal{M}$ is *strategy-proof* if for every preference profile $\succ$, and for each doctor $d \in D$, there is no $\hat{\succ}_d$ such that $\mathcal{M}(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}}) \succ_d \mathcal{M}(\succ)$.

One set of mechanisms of particular importance is the class of *cumulative offer mechanisms*. In a cumulative offer mechanism $\mathcal{C}^{\vdash}$, doctors propose contracts according to a strict ordering $\vdash$ of the elements of X. In every step, some doctor who does not currently have a contract held by any hospital proposes his most-preferred contract that has not yet been proposed; then, each hospital chooses its most-preferred set of contracts according to its choice function, and holds this set until the next step. When multiple doctors are available to propose in the same step, the doctor who actually proposes is determined by the ordering $\vdash$. The mechanism terminates when no doctor is able to propose; at that point, each hospital is assigned the set of contracts it is holding. (We describe cumulative offer mechanisms more formally in Appendix A.)

II. Stable and Strategy-Proof Matching

In recent work (Hatfield et al., 2016), we characterized the conditions on hospital preferences needed to guarantee the existence of stable and strategy-proof matching mechanisms. Moreover, we showed that when stable and strategy-proof matching *is* possible, the outcome of any stable and strategy-proof matching mechanism coincides with that of a cumulative offer mechanism and, in fact, the outcomes of all cumulative offer mechanisms coincide.

In a range of matching settings, including many-to-one matching without contracts and many-to-many matching with contracts, stable and strategy-proof matching hinges upon two conditions: *substitutability* and *size monotonicity*. A choice function is *substitutable* if no two contracts x and z are "complementary" in the sense that gaining access to x makes z more attractive. That is, C^h is *substitutable* if for all contracts x and z and sets of contracts Y, if $z \notin C^h(Y \cup \{z\})$, then $z \notin C^h(\{x\} \cup Y \cup \{z\})$. Substitutability is equivalent to monotonicity of the rejection function: C^h is substitutable if and only if we have $R^h(Y) \subseteq R^h(Z)$ for all sets of contracts Y and Z such that $Y \subseteq Z$. The choice function of hospital $h \in H$ is *size monotonic* if h chooses weakly more contracts whenever the set of available contracts expands, i.e., if for all contracts z and sets of contracts Y, we have $|C^h(Y)| \leq |C^h(Y \cup \{z\})|$.[1]

In fact, for a cumulative offer mechanism to be stable and strategy-proof, substitutability and size monotonicity need only hold during the running of the mechanism itself; however, in that case, we also need to rule out within-hospital manipulation. Formally, an *offer process for h* is a finite sequence of distinct contracts $(x^1, \ldots, x^M) = \mathbf{x}$ such that $x^m \in X_h$ for all $m = 1, \ldots, M$. The offer process $\mathbf{x}$ for h is *observable* if, for all $m = 1, \ldots, M$, we have that $\mathsf{d}(x^m) \notin \mathsf{d}(C^h(\{x^1, \ldots, x^{m-1}\}))$; roughly,

[1]Size monotonicity is often called the *Law of Aggregate Demand* (Hatfield and Milgrom, 2005).

an offer process is observable if it can occur as a sequence of proposals under a cumulative offer mechanism. For the offer process $\mathbf{x} = (x^1, \ldots, x^M)$, we let $\mathsf{c}(\mathbf{x})$ denote the set of contracts "offered" in $\mathbf{x}$, i.e., $\mathsf{c}(\mathbf{x}) \equiv \{x^1, \ldots, x^M\}$.

DEFINITION 1: *The choice function C^h* exhibits an observable violation of substitutability *if there exists an observable offer process $(x^1, \ldots, x^M)$ for h such that $R^h(\{x^1, \ldots, x^{M-1}\}) \smallsetminus R^h(\{x^1, \ldots, x^M\}) \neq \varnothing$. The choice function C^h is* observably substitutable *if it does not exhibit an observable violation of substitutability.*

DEFINITION 2: *The choice function C^h* exhibits an observable violation of size monotonicity *if there exists an observable offer process $(x^1, \ldots, x^M)$ for h such that $|C^h(\{x^1, \ldots, x^M\})| < |C^h(\{x^1, \ldots, x^{M-1}\})|$. The choice function C^h is* observably size monotonic *if it does not exhibit an observable violation of size monotonicity.*

DEFINITION 3: *The choice function C^h is* manipulable via contractual terms (absent other hospitals) *if there is a strict ordering $\vdash$ of X_h, a preference profile $\succ$ for doctors under which only contracts with h are acceptable, a doctor d, and a preference relation $\hat{\succ}_d$ for d under which only contracts with h are acceptable such that $\mathcal{C}^{\vdash}(\hat{\succ}_d, \succ_{-d}) \succ_d \mathcal{C}^{\vdash}(\succ)$.*

The preceding three conditions exactly characterize when stable and strategy-proof matching can be guaranteed (Hatfield et al., 2016); we state the sufficiency result here.

THEOREM 1 (Hatfield et al., 2016): *If each hospital's choice function C^h is observably substitutable and observably size monotonic, and is not manipulable via contractual terms, then cumulative offer mechanisms are stable and strategy-proof.*

III. Matching with Flexible Allotments

We now introduce a model of hospital choice in which each hospital has a set of divisions and a flexible allotment of capacities to those divisions that varies as a function of the set of contracts available. We model a hospital h as having a set of *divisions* $S = \{1, \ldots, \bar{s}\}$.[2] Each division $s \in S$ has an *extended choice function* $C^s : \wp(X) \times \mathbb{Z}_{\geq 0} \to \wp(X_h)$ that specifies the contracts s chooses when given a set of contract offers and an *allotment* of positions to fill.[3] We require that each division s never chooses more contracts than its allotment—that is, for a set of contracts $Y \subseteq X_h$ and allotment a, we must have $|C^s(Y; a)| \leq a$.

We also model the hospital h as having an *allotment function* $q : \wp(X_h) \to (\mathbb{Z}_{\geq 0})^S$ that determines how many positions are allocated to each division, given the available set of contracts.[4] For each division $s \in S$, we let $C^s(\cdot; \infty) \equiv C^s(\cdot; |X_h|)$ be the *unconstrained choice function for s*; this function encodes the "true preferences" of s that arise when the allotment constraint does not bind.

For each division $s \in S$, we require that C^s satisfies the classical substitutability, size monotonicity, and irrelevance of rejected contracts conditions when the allotment is held fixed. That is, we require:

1) For all allotments a, and for all contracts $x, z \in X$ and sets of contracts $Y \subseteq X$, if $z \notin C^s(Y \cup \{z\}; a)$, then we have that $z \notin C^s(\{x\} \cup Y \cup \{z\}; a)$.
2) For all allotments a, and for all sets of contracts $Y, Z \subseteq X$ such that $Y \subseteq Z$, we have that $|C^s(Y; a)| \leq |C^s(Z; a)|$.
3) For all allotments a, and for all contracts $z \in X$ and sets of contracts $Y \subseteq X$, if $z \notin C^s(Y \cup \{z\}; a)$, then we have that $C^s(Y; a) = C^s(Y \cup \{z\}; a)$.

We also impose two conditions on how extended choice functions respond to changes in allotments for each division $s \in S$:

1) Extended choice functions are *monotone with respect to the allotment*, i.e., for all sets of contracts $Y \subseteq X$ and allotments $\check{a}$ and $\hat{a}$ such that $\check{a} \leq \hat{a}$, we have that $C^s(Y; \check{a}) \subseteq C^s(Y; \hat{a})$. Intuitively, the extended choice function of division s is monotonic with respect to the allotment if whenever s chooses a contract under an allotment $\check{a}$, s still chooses that contract when given a larger allotment $\hat{a} \geq \check{a}$.

[2]Throughout, for notational simplicity, we suppress the dependence of C^h's primitives on the hospital h.

[3]Here, the $\wp(X)$ denotes the power set of X.

[4]Abusing notation slightly, for a set of contracts $Y \subseteq X$, we let $q(Y) \equiv q(Y_h)$.

2) Extended choice functions are *conditionally acceptant*, i.e., for all sets of contracts $Y \subseteq X$ and allotments a, if $a \leq |C^s(Y;\infty)|$, then $|C^s(Y;a)| = a$.

Given the extended choice functions C^s for divisions $s \in S$, as well as the allotment function q, we calculate the choice of h from a set of contracts $Y \subseteq X$ according to the following *choice procedure*:

Step 0**:** Initialize the set of *available contracts* as $Y_{\rightsquigarrow 1} \equiv Y$.

Step s**:** Division s chooses up to $q^s(Y)$ contracts from the set of contracts $Y_{\rightsquigarrow s}$ still available; then all contracts with doctors chosen by s are made unavailable to other divisions. That is, let $G^s \equiv C^s(Y_{\rightsquigarrow s}; q^s(Y))$ and then let the set of contracts available to division $s+1$ be given by $Y_{\rightsquigarrow s+1} \equiv Y_{\rightsquigarrow s} \smallsetminus \{y \in Y_{\rightsquigarrow s} : \mathsf{d}(y) \in G^s\}$.

Step $\bar{s}+1$**:** The choice of h is then given by $C^h(Y) \equiv \cup_{s \in S} G^s$.

Throughout, we use $Y_{\rightsquigarrow s}$ to denote the set of contracts available to division s at its step in the computation of $C^h(Y)$. Similarly, for an offer process $\mathbf{x}$, we let $\mathsf{c}_{\rightsquigarrow s}(\mathbf{x})$ denote the set of contracts available to division s at its step in the computation of $C^h(\mathsf{c}(\mathbf{x}))$.

We assume throughout that the allotment function *does not depend on irrelevant contracts*, i.e., for every set of contracts Y, if $Z \subseteq R^h(Y)$, then $q(Y \smallsetminus Z) = q(Y)$. We also impose throughout three substantive conditions on the allotment function q:

1) The allotment function does not *not observably grant excess positions.* That is, for every observable offer process $\mathbf{x}$ and every division $s \in S$, we have

$$q^s(\mathsf{c}(\mathbf{x})) = |C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x})))|,$$

so that each division s exactly fills its allotment of slots out of $\mathsf{c}(\mathbf{x})$.

2) The allotment function is *single-peaked across observable offer processes.* That is, for any observable offer processes $\mathbf{x}$ and $\mathbf{y}$ such that $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\mathbf{y})$, the allotment of division s for $\mathbf{y}$ strictly increases from the allotment for $\mathbf{x}$ only if division s was unconstrained under $\mathbf{x}$—formally, $q^s(\mathsf{c}(\mathbf{y})) > q^s(\mathsf{c}(\mathbf{x}))$ only if $|C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x});\infty)| = q^s(\mathbf{x})$.[5]

3) The allotment function is *monotone in aggregate across observable offer processes.* That is, for any observable offer processes $\mathbf{x}$ and $\mathbf{y}$ such that $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\mathbf{y})$, we have that $\sum_{s\in S} q^s(\mathsf{c}(\mathbf{x})) \leq \sum_{s \in S} q^s(\mathsf{c}(\mathbf{y}))$.

If a hospital h has a choice function that can be represented by a set of divisions S and an allotment function q such that

- each division's extended choice function is substitutable and size monotonic, satisfies the irrelevance of rejected contracts condition for any allotment, and, moreover, is monotonic with respect to the allotment and conditionally acceptant, and
- the allotment function neither depends on irrelevant contracts nor observably grants excess positions, and is both single-peaked and monotone in aggregate across observable offer processes,

then we say that h has a *multi-division choice function with flexible allotments.* Roughly, multi-division choice functions with flexible allotments model settings in which each hospital, upon hiring a candidate into a given division, may have to reduce the number of positions it allots to other divisions (e.g., because of hospital-wide budget constraints).

Multi-division choice functions with flexible allotments satisfy the three key conditions we introduced in Section II.

THEOREM 2: *If hospital h has a multi-division choice function with flexible allotments, then the choice function of h is observably substitutable, observably size monotonic, and non-manipulable via contractual terms, and also satisfies the irrelevance of rejected contracts condition.*[6]

Combined with Theorem 1, Theorem 2 shows that matching with flexible allotments allows for stable and strategy-proof matching.

[5] We refer to this property as a form of "single-peakedness" because, when combined with the no excess positions condition, it implies that when an observable offer process is expanded, the allotment for s is first (weakly) increasing and then (weakly) decreasing.

[6] The proof of this result is in Appendix B.

COROLLARY 1: *If every hospital has a multi-division choice function with flexible allotments, then cumulative offer mechanisms are stable and strategy-proof.*

To obtain Corollary 1, we in fact need the full generality of the sufficient conditions of Hatfield et al. (2016), as multi-division choice functions with flexible allotments fail in general to satisfy the weakest prior conditions known to ensure stable and strategy-proof matching.[7]

IV. Discussion

Matching with flexible allotments allows us to model firms with cross-division hiring constraints. Moreover, the flexible allotments framework may be useful in understanding stable and strategy-proof matching in settings with distributional constraints (see, e.g., Sönmez and Switzer (2013) and Aygün and Turhan (2017)). For a start, we show in Appendix D that our model nests the baseline *regional caps* model of Kamada and Kojima (2015, 2017); moreover, our framework straightforwardly embeds models of matching with reserves and quotas, such as the *slot-specific priorities* framework of Kominers and Sönmez (2016).

Our work here illustrates the value of clearly mapping when stable and strategy-proof matching is possible—our theory of observable substitutability allows us to efficiently demonstrate that matching with flexible allotments admits stable and strategy-proof matching, even though it falls outside the purview of previous work.

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[7] In Appendix C, we show that Example 2 of Hatfield et al. (2016) can be expressed in our model; this proves that there exists a multi-division choice function with flexible allotments that does *not* satisfy the substitutable completability condition of Hatfield and Kominers (2016).

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Online Appendix

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A. The Cumulative Offer Mechanism

In this appendix, we formally define the cumulative offer mechanisms described in Section I.B. For any preference profile $\succ$, the outcome of the cumulative offer mechanism according to the strict ordering $\vdash$ of the elements of X, denoted by $\mathcal{C}^{\vdash}(\succ)$, is determined by the *cumulative offer process with respect to* $\vdash$ *and* $\succ$ as follows:

Step 0: Initialize the set of contracts *available to the hospitals (at the end of step* 0*)* as $A^0 = \varnothing$, and initialize the set of contracts *held by hospitals (at the end of step* 0*)* as $L^0 = \varnothing$.

Step $k \geq 1$: Consider the set

$$U^k \equiv \{x \in X \smallsetminus A^{k-1} : \mathsf{d}(x) \notin \mathsf{d}(L^{k-1}) \text{ and } \nexists z \in (X_{\mathsf{d}(x)} \smallsetminus A^{k-1}) \cup \{\emptyset\} \text{ such that } z \succ_{\mathsf{d}(x)} x\},$$

which consists of those contracts *not yet available to hospitals* that are most-preferred by doctors who do not have contracts currently held by any hospital.

- If U^k is not empty, we let y^k be the highest-ranked element of U^k according to $\vdash$. Doctor $\mathsf{d}(y^k)$ *proposes* y^k, making it available to $\mathsf{h}(y^k)$. We update the set of *available* contracts to $A^k = A^{k-1} \cup \{y^k\}$; then, the hospitals *hold* $L^k = \cup_{h \in H} C^h(A^k)$, and we proceed to step $k+1$.
- If U^k is empty, then the cumulative offer process terminates and the outcome is given by L^{k-1}.[1]

We let K denote the last proposal step of the cumulative offer process with respect to $\vdash$ and $\succ$, and call A^K the set of contracts *observed* in the cumulative offer process with respect to $\vdash$ and $\succ$.

Note that the sequence $(y^1, \ldots, y^K)$ of contracts proposed in the cumulative offer process with respect to $\vdash$ and $\succ$ is, in fact, an observable offer process (for all hospitals h). Indeed, for each proposed contract $y^k \in U^k$, we have

$$\mathsf{d}(y^k) \notin \mathsf{d}(L^{k-1}) = \mathsf{d}(\cup_{h \in H} C^h(A^{k-1})) = \mathsf{d}(\cup_{h \in H} C^h(\{y^1, \ldots, y^{k-1}\})),$$

as is required for observability. Even so, however, without further assumptions on hospitals' choice functions, the outcome of a cumulative offer process need not be feasible, i.e., it might be the case that $L^K = \cup_{h \in H} C^h(A^K)$ contains more than one contract with a given doctor.

B. Proof of Theorem 2

In this appendix, we prove our main result, Theorem 2.

B1. Preliminaries

We adapt the notation of Hatfield et al. (2016). For an offer process $\mathbf{x} = (x^1, \ldots, x^M)$, we denote the offer process $(x^1, \ldots, x^m)$ by $\mathbf{x}^m$.[2] For an offer process $\mathbf{x} = (x^1, \ldots, x^M)$, we (somewhat informally) think of $\mathbf{x}$ as representing a sequence of proposals by doctors in a cumulative offer process (see Appendix A). With that intuition, we may think of hospital h as evaluating the set of contracts $\mathsf{c}(\mathbf{x}^m)$ "available" at "step" m of the offer process; under a multi-division choice function with flexible allotments C^h, this implicitly involves each division $s \in S$ evaluating $\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m)$, the subset of $\mathsf{c}(\mathbf{x}^m)$ that s has the opportunity to consider in the computation of

[1] Note that if U^k is empty, all doctors who currently do not have a contract on hold have already proposed all the contracts they find acceptable.

[2] In particular, we use the convention that $\mathbf{x}^0$ represents the empty sequence.

$C^h(\mathsf{c}(\mathbf{x}^m))$. In the sequel, it is useful for us to track the full set of contract offers that are ever considered by s in this way by step m; we call this set, $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}) \equiv \cup_{m\leq M}\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m)$, the set of contracts that division s has access to at some step of the offer process.

Additionally, for each division $s \in S$ we define an *extended rejection function*

$$R^s(Y;a) \equiv Y \smallsetminus C^s(Y;a),$$

which specifies the contracts s does not choose when given a set of contract offers Y and an allotment of positions a. We note that when $C^s(\cdot;a)$ is substitutable, $R^s(\cdot;a)$ is isotone.

B2. Observable Substitutability

We show first that the choice of division s at the end of an observable offer process $\mathbf{x}$, i.e., $C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x})))$, coincides with the choice by s from the union of its opportunity sets along the observable offer process, i.e., $C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x})))$. Moreover, for any observable offer process $\mathbf{y}$ such that every contract offered during $\mathbf{x}$ is offered during $\mathbf{y}$ (i.e., $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\mathbf{y})$) we have that any contract rejected by division s when $\mathsf{c}(\mathbf{x})$ is available to h is also rejected by s when $\mathsf{c}(\mathbf{y})$ is available to h.

CLAIM 1: *If $\mathbf{x}$ is an observable offer process, then for each division $s \in S$,*

(B1) $$C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x}))) = C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x}))).$$

Furthermore, if $\mathbf{y}$ is an observable offer process such that $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\mathbf{y})$, and $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}) \subseteq \mathsf{f}_{\rightsquigarrow s}(\mathbf{y})$ for some division $s \in S$, then

(B2) $$R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x}))) \subseteq R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{y}); q^s(\mathsf{c}(\mathbf{y}))).$$

PROOF:

We fix an observable offer process $\mathbf{x} = (x^1, \dots, x^M)$. We proceed by induction on pairs (m, s) in the order

$$(1,1),(1,2),\dots,(1,\bar{s}),(2,1),(2,2),\dots,(2,\bar{s}),\dots,(M,1),(M,2),\dots,(M,\bar{s}),$$

showing at each step that

(B3a) $$C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) = C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))),$$

(B3b) $$C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) \cup [\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m) \smallsetminus \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1})],$$

(B3c) $$C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) \subseteq \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m),$$

(B3d) $$R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{y}); q^s(\mathsf{c}(\mathbf{y})))$$

for all observable offer processes $\mathbf{y}$ such that $\mathsf{c}(\mathbf{x}^m) \subseteq \mathsf{c}(\mathbf{y})$ and divisions s such that $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) \subseteq \mathsf{f}_{\rightsquigarrow s}(\mathbf{y})$. Taking $m = M$ then provides the desired results via (B3a) and (B3d).

Base Case(s). For $m = 1$, conditions (B3a), (B3b), and (B3c) follow immediately for all $s \in S$. Condition (B3d) follows using the same argument as we use in the general (m, s) case *infra.*

Inductive Step. We now show that (B3) holds for (m, s) if (B3) holds for

- every pair $(\hat{m}, t)$ with $\hat{m} < m$ and $t \in S$, as well as
- every pair (m, t) with $t < s$.

Condition (B3c)**.** First, we note that the result is immediate if $s = 1$, as $\mathsf{c}_{\rightsquigarrow 1}(\mathbf{x}^{m-1}) = \mathsf{c}(\mathbf{x}^{m-1}) \subseteq \mathsf{c}(\mathbf{x}^m) = \mathsf{c}_{\rightsquigarrow 1}(\mathbf{x}^m)$. Thus, we consider any contract z chosen by division $s > 1$ at step $m - 1$, that is, any $z \in C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1})))$. Under the choice procedure defining C^h, since $z \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1})$, it must be that no contract with $\mathsf{d}(z)$ is chosen by any division $t < s$ at step $m-1$, i.e.,

$$[\mathsf{c}(\mathbf{x}^{m-1})]_{\mathsf{d}(z)} \subseteq \bigcap_{t<s} R^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{m-1}); q^t(\mathsf{c}(\mathbf{x}^{m-1}))).$$

Moreover, conditions (B3a) and (B3b) together imply that for each $t < s$,

$$C^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^m); q^t(\mathsf{c}(\mathbf{x}^m))) \subseteq C^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{m-1}); q^t(\mathsf{c}(\mathbf{x}^{m-1}))) \cup [\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^m) \smallsetminus \mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{m-1})].$$

Hence, we have

$$[\mathsf{c}(\mathbf{x}^{m-1})]_{\mathsf{d}(z)} \subseteq \bigcap_{t<s} R^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^m); q^t(\mathsf{c}(\mathbf{x}^m))), \tag{B4}$$

i.e., it must be that no contract with $\mathsf{d}(z)$—except possibly x^m—is chosen by any division $t < s$ at step m. But since $\mathbf{x}^m$ is observable and $z \in C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1})))$, it must be that $\mathsf{d}(z) \neq \mathsf{d}(x^m)$. Thus, (B4) implies

$$[\mathsf{c}(\mathbf{x}^m)]_{\mathsf{d}(z)} \subseteq \bigcap_{t<s} R^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^m); q^t(\mathsf{c}(\mathbf{x}^m))),$$

that is, no contract with $\mathsf{d}(z)$ is chosen by any division $t < s$ at step m. Thus, any contract proposed by $\mathsf{d}(z)$—and, in particular, z—is available to s at step m. Therefore, since z was an arbitrary element of $C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1})))$, we have that $C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) \subseteq \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m)$, as desired.

Condition (B3b)**.** Taking $\mathbf{y} = \mathbf{x}^m$ in condition (B3d) for $(m-1, s)$, we obtain

$$R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) \subseteq R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))),$$

which implies that

$$C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) \cup [\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) \smallsetminus \mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{m-1})].$$

Since $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) = \cup_{n \le m} \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^n)$ and $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{m-1}) = \cup_{n \le m-1} \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^n)$ by definition, we have

$$\begin{aligned}
\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) \smallsetminus \mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{m-1}) &= [\cup_{n \le m} \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^n)] \smallsetminus [\cup_{n \le m-1} \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^n)] \\
&= [\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m) \smallsetminus [\cup_{n \le m-1} \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^n)]] \cup [[\cup_{n \le m-1} \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^n)] \smallsetminus [\cup_{n \le m-1} \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^n)]] \\
&= [\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m) \smallsetminus [\cup_{n \le m-1} \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^n)]] \cup \varnothing \\
&\subseteq \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m) \smallsetminus \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}).
\end{aligned}$$

Combining the two immediately preceding expressions yields

$$C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) \cup [\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m) \smallsetminus \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1})]. \tag{B5}$$

Now, since condition (B3a) holds for $(m-1, s)$, we have that

$$C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) = C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))). \tag{B6}$$

Combining (B5) and (B6) implies that

$$C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) \cup [\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m) \smallsetminus \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1})],$$

as desired.

Condition (B3a)**.** Condition (B3b) for (m,s) implies that

$$C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) \cup [\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m) \smallsetminus \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1})];$$

condition (B3c) implies that $C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{m-1}); q^s(\mathsf{c}(\mathbf{x}^{m-1}))) \subseteq \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m)$, and so

$$C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m).$$

Since $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) \supseteq \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m)$, the fact that the extended choice function of division s satisfies the irrelevance of rejected contracts condition then implies that

$$C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) = C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))),$$

as desired.

Condition (B3d)**.** Suppose that $\mathbf{y}$ is an observable offer process such that $\mathsf{c}(\mathbf{x}^m) \subseteq \mathsf{c}(\mathbf{y})$ and $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) \subseteq \mathsf{f}_{\rightsquigarrow s}(\mathbf{y})$ for some $s \in S$. There are two cases to consider:

Case 1: $q^s(\mathsf{c}(\mathbf{x}^m)) \geq q^s(\mathsf{c}(\mathbf{y}))$**.** Since $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) \subseteq \mathsf{f}_{\rightsquigarrow s}(\mathbf{y})$, the substitutability of the extended choice function of division s implies that $R^s(\mathsf{c}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq R^s(\mathsf{c}(\mathbf{y}); q^s(\mathsf{c}(\mathbf{x}^m)))$. Since $q^s(\mathsf{c}(\mathbf{x}^m)) \geq q^s(\mathsf{c}(\mathbf{y}))$ and the extended choice function of s is monotonic with respect to the allotment, we have $R^s(\mathsf{c}(\mathbf{y}); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq R^s(\mathsf{c}(\mathbf{y}); q^s(\mathsf{c}(\mathbf{y})))$ and so

$$R^s(\mathsf{c}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq R^s(\mathsf{c}(\mathbf{y}); q^s(\mathsf{c}(\mathbf{y}))), \tag{B7}$$

as desired.

Case 2: $q^s(\mathsf{c}(\mathbf{x}^m)) < q^s(\mathsf{c}(\mathbf{y}))$**.** We first show that

$$C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) = C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty). \tag{B8}$$

To show (B8), we suppose the contrary—i.e., that $C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \neq C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$—and seek a contradiction. Now, if $C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \neq C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$, then, as the extended choice function of s is monotonic with respect to the allotment, we must have $C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subsetneq C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$. We let $z \in C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty) \smallsetminus C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m)))$. As $z \in \mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m)$, there must exist some largest $\bar{m} \leq m$ such that $z \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{\bar{m}})$. There are then two subcases to consider:

If $\bar{m} = m$**,** then since condition (B3a) holds for (m,s), we must have that $C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) = C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m)))$. As $z \notin C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m)))$ and $z \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m)$, we then have that

$$z \in R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))).$$

As the allotment function is single-peaked across observable offer processes and $q^s(\mathsf{c}(\mathbf{x}^m)) < q^s(\mathsf{c}(\mathbf{y}))$, we have that $R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) = R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$; in particular, $z \in R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$. Thus, as the extended choice function of s is substitutable, we have $z \in R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$, contradicting our assumption that $z \in C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$.

If $\bar{m} < m$**,** then $z \notin \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{\bar{m}+1})$, as we chose $\bar{m}$ to be the largest $\bar{m} \leq m$ such that $z \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{\bar{m}})$. Thus, $z \in R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{\bar{m}}); q^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{\bar{m}})))$, as otherwise condition (B3c) is violated. As the allotment function is single-peaked across observable offer processes and $q^s(\mathsf{c}(\mathbf{x}^m)) < q^s(\mathsf{c}(\mathbf{y}))$, we have that $R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{\bar{m}}); q^s(\mathsf{c}(\mathbf{x}^{\bar{m}}))) = R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{\bar{m}}); \infty)$; in particular, $z \in R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{\bar{m}}); \infty)$. Thus, as the extended choice function of s is substitutable, $z \in R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{\bar{m}}); \infty)$. Then, again as the extended choice function of s is substitutable, we must have $z \in R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$, contradicting our assumption that $z \in C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$.

The preceding argument shows (B8), which implies that $R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) = R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); \infty)$. The substitutability of the extended choice function of s then implies that $R^s(\mathsf{f}_{\rightsquigarrow s}(\mathsf{c}(\mathbf{x}^m)); q^s(\infty)) \subseteq R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{y}); \infty)$, and so

$$R^s(\mathsf{f}_{\rightsquigarrow s}(\mathsf{c}(\mathbf{x}^m)); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{y}); \infty).$$

Finally, the monotonicity of the extended choice function of s with respect to allotment (i.e., $R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{y}); \infty) \subseteq R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{y}); q^s(\mathsf{c}(\mathbf{y}))))$ implies (B7) in this case too, completing our induction.

We now show that the choice function of h is observably substitutable.

CLAIM 2: *The choice function C^h is observably substitutable.*

PROOF:
We consider an observable offer process $\mathbf{x} = (x^1, \ldots, x^M)$ and let $y \in R^h(\{x^1, \ldots, x^{M-1}\})$. Under the choice procedure defining C^h, since $y \in R^h(\{x^1, \ldots, x^{M-1}\})$, we have that $y \in R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}); q^s(\mathsf{c}(\mathbf{x}^{M-1})))$ for each $s \in S$. The substitutability of the extended choice function of each $s \in S$ then implies that $y \in R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{M-1}); q^s(\mathsf{c}(\mathbf{x}^{M-1})))$ for each $s \in S$. Since $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{M-1}) \subseteq \mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^M)$ by construction, (B2) of Claim 1 implies that $y \in R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^M); q^s(\mathsf{c}(\mathbf{x}^M)))$ for each $s \in S$. Thus, $y \notin C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^M); q^s(\mathsf{c}(\mathbf{x}^M)))$ for each $s \in S$; (B1) of Claim 1 then implies that $y \notin C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^M); q^s(\mathsf{c}(\mathbf{x}^M)))$ for each $s \in S$. Thus, under the choice procedure defining C^h, we have that $y \in R^h(\{x^1, \ldots, x^M\})$, as desired.

B3. Observable Size Monotonicity

Next, we show that the choice function C^h is observably size monotonic across observable offer processes.

CLAIM 3: *The choice function C^h is observably size monotonic.*

PROOF:
Consider any two observable offer process $\mathbf{x}$ and $\hat{\mathbf{x}}$ such that $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$. As the allotment function does not observably grant excess positions, we have that $q^s(\mathsf{c}(\mathbf{x})) = |C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x})))|$ for each division $s \in S$, which implies that $\sum_{s\in S} q^s(\mathsf{c}(\mathbf{x})) = \sum_{s\in S} |C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x})))|$. Similarly, we have that $\sum_{s\in S} q^s(\mathsf{c}(\hat{\mathbf{x}})) = \sum_{s\in S} |C^s(\mathsf{c}_{\rightsquigarrow s}(\hat{\mathbf{x}}); q^s(\mathsf{c}(\hat{\mathbf{x}})))|$. Now, as the allotment function is monotone in aggregate across observable offer processes, we have that

$$\sum_{s\in S} q^s(\mathsf{c}(\mathbf{x})) \leq \sum_{s\in S} q^s(\mathsf{c}(\hat{\mathbf{x}}));$$

hence, we have

$$|C^h(\mathsf{c}(\mathbf{x}))| = \sum_{s\in S} |C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x})))| \leq \sum_{s\in S} |C^s(\mathsf{c}_{\rightsquigarrow s}(\hat{\mathbf{x}}); q^s(\mathsf{c}(\hat{\mathbf{x}})))| = |C^h(\mathsf{c}(\hat{\mathbf{x}}))|,$$

so the choice function C^h is observably size monotonic, as desired.

B4. (Non-)Manipulability via Contractual Terms

We now establish that C^h is non-manipulable via contractual terms. Consider an arbitrary doctor $d \in D$, and let $z^0, z^1, \ldots, z^N$ be an arbitrary sequence of contracts in X_d. Fix a profile of preferences $\succ_{D\smallsetminus\{d\}}$ for all other doctors, and let $\succ_d$ and $\hat{\succ}_d$ be given by

$$\succ_d : z^1 \succ_d \ldots \succ_d z^N, \tag{B9}$$

$$\hat{\succ}_d : z^0 \hat{\succ}_d z^1 \hat{\succ}_d \ldots \hat{\succ}_d z^N. \tag{B10}$$

We fix an ordering $\vdash$ over the set of contracts X, and let $\mathbf{x} = (x^1, \ldots, x^M)$ be the observable offer process induced by the cumulative offer mechanism with ordering $\vdash$ under the preferences $(\succ_d, \succ_{D\smallsetminus\{d\}})$ when only hospital h is present. Similarly, let $\hat{\mathbf{x}} = (\hat{x}^1, \ldots, \hat{x}^{\hat{M}})$ be the observable offer process induced by the cumulative offer mechanism with ordering $\vdash$ under the preferences $(\hat{\succ}_d, \succ_{D\smallsetminus\{d\}})$ when only hospital h is present. We first establish the following claim.

CLAIM 4: *If $z^0 \notin C^h(\mathsf{c}(\hat{\mathbf{x}}))$, then*

$$R^h(\mathsf{c}(\mathbf{x})) \subseteq R^h(\mathsf{c}(\hat{\mathbf{x}})) \tag{B11}$$

and, for all $s \in S$, we have that

$$\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}) \subseteq \mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}}). \tag{B12}$$

PROOF:

We proceed by induction on pairs (m, s) in the order

$$(1,1), (1,2), \ldots, (1,\bar{s}), (2,1), (2,2), \ldots, (2,\bar{s}), \ldots, (M,1), (M,2), \ldots, (M,\bar{s}),$$

showing at each step that

$$\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) \subseteq \mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}}), \tag{B13a}$$

$$R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq R^s(\mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}}); q^s(\mathsf{c}(\hat{\mathbf{x}}))). \tag{B13b}$$

Once we have completed our induction, (B12) follows from taking $m = M$.

For the base case $(1,1)$, it must be the case that x^1 is either the highest-ranked contract by some doctor $\mathsf{d}(x^1) \neq d$ or $x^1 = z^1$. In the former case, x^1 must be offered at some step in the offer process $\hat{\mathbf{x}}$, as it is the favored contract of doctor $\mathsf{d}(x^1)$. In the latter case, since z^0 is rejected by h by assumption, d must offer his second-favorite contract under $\hat{\succ}_d$, i.e., x^1, at some step in the offer process $\hat{\mathbf{x}}$. Hence, in both cases we have that $x^1 \in \mathsf{c}(\hat{\mathbf{x}})$. Since $\mathsf{f}_{\rightsquigarrow 1}(\hat{\mathbf{x}}) = \mathsf{c}(\hat{\mathbf{x}})$, we obtain that $\mathsf{f}_{\rightsquigarrow 1}(\mathbf{x}^1) \subseteq \mathsf{f}_{\rightsquigarrow 1}(\hat{\mathbf{x}})$. Then, condition (B2) of Claim 1 implies that $R^1(\mathsf{f}_{\rightsquigarrow 1}(\mathbf{x}^1); q^1(\mathsf{c}(\mathbf{x}^1))) \subseteq R^1(\mathsf{f}_{\rightsquigarrow 1}(\hat{\mathbf{x}}); q^1(\mathsf{c}(\hat{\mathbf{x}})))$.

We now show that (B13) holds for (m, s) if (B13) holds for

- every pair $(\bar{m}, t)$ with $\bar{m} < m$ and $t \in S$, as well as
- every pair (m, t) with $t < s$.

We show first that (B13) holds for $(m, 1)$ given that (B13) holds for every pair $(\bar{m}, t)$ with $\bar{m} < m$ and $t \in S$. By the inductive assumption (B13b) for pairs $(m-1, t)$, we have

$$R^t(\mathsf{f}_{\rightsquigarrow t}(\mathbf{x}^{m-1}); q^t(\mathsf{c}(\mathbf{x}^{m-1}))) \subseteq R^t(\mathsf{f}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}})))$$

for all t. As $\mathbf{x}^m$ is observable, we have that $\{x^1, \ldots, x^{m-1}\}_{\mathsf{d}(x^m)} \subseteq R^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{m-1}); q^t(\mathsf{c}(\mathbf{x}^{m-1})))$ for all t. Moreover, as $R^t(\cdot; q^t(\mathsf{c}(\mathbf{x}^{m-1})))$ is substitutable, we have $R^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{m-1}); q^t(\mathsf{c}(\mathbf{x}^{m-1}))) \subseteq R^t(\mathsf{f}_{\rightsquigarrow t}(\mathbf{x}^{m-1}); q^t(\mathsf{c}(\mathbf{x}^{m-1})))$, so that

$$\{x^1, \ldots, x^{m-1}\}_{\mathsf{d}(x^m)} \subseteq R^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{m-1}); q^t(\mathsf{c}(\mathbf{x}^{m-1}))) \subseteq R^t(\mathsf{f}_{\rightsquigarrow t}(\mathbf{x}^{m-1}); q^t(\mathsf{c}(\mathbf{x}^{m-1}))).$$

Hence, given that $R^t(\mathsf{f}_{\rightsquigarrow t}(\mathbf{x}^{m-1}); q^t(\mathsf{c}(\mathbf{x}^{m-1}))) \subseteq R^t(\mathsf{f}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}})))$ for all t, we find that $\{x^1, \ldots, x^{m-1}\}_{\mathsf{d}(x^m)} \subseteq R^t(\mathsf{f}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}})))$ for all t.

By Condition (B1) of Claim 1, we see that there is an $M' \leq \hat{M}$ such that $\{x^1, \ldots, x^{m-1}\}_{\mathsf{d}(x^m)} \subseteq R^t(\mathsf{c}_{\rightsquigarrow t}(\hat{\mathbf{x}}^{M'}); q^t(\mathsf{c}(\hat{\mathbf{x}}^{M'})))$ for all t. Since $\hat{\mathbf{x}}$ is observable and represents all the offers made under the cumulative offer process for $(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}})$, there must exist some step $\hat{m}$ at which x^m is proposed in $\hat{\mathbf{x}}$; hence, recalling that $\mathsf{c}(\hat{\mathbf{x}}) = \mathsf{f}_{\rightsquigarrow 1}(\hat{\mathbf{x}})$, we have that

$$x^m \in \mathsf{c}(\hat{\mathbf{x}}) = \mathsf{f}_{\rightsquigarrow 1}(\hat{\mathbf{x}}). \tag{B14}$$

Additionally, by the inductive assumption (B13a) for $m-1$ and $s=1$, we have that

$$\mathsf{c}(\mathbf{x}^{m-1}) = \mathsf{f}_{\rightsquigarrow 1}(\mathbf{x}^{m-1}) \subseteq \mathsf{f}_{\rightsquigarrow 1}(\hat{\mathbf{x}}). \tag{B15}$$

Moreover, recalling that $\mathsf{f}_{\rightsquigarrow 1}(\mathbf{x}^m) = \mathsf{c}(\mathbf{x}^m)$ and $\mathsf{c}(\mathbf{x}^m) = \{x^m\} \cup \mathsf{c}(\mathbf{x}^{m-1})$, we have that

$$\mathsf{f}_{\rightsquigarrow 1}(\mathbf{x}^m) = \mathsf{c}(\mathbf{x}^m) = \{x^m\} \cup \mathsf{c}(\mathbf{x}^{m-1}). \tag{B16}$$

Combining (B14) and (B15) with (B16) then implies that $\mathsf{f}_{\rightsquigarrow 1}(\mathbf{x}^m) \subseteq \mathsf{f}_{\rightsquigarrow 1}(\hat{\mathbf{x}})$, which is exactly condition (B13a) for $(m, 1)$. Applying condition (B2) of Claim 1 yields that $R^1(\mathsf{f}_{\rightsquigarrow 1}(\mathbf{x}^m); q^1(\mathsf{c}(\mathbf{x}^m))) \subseteq R^1(\mathsf{f}_{\rightsquigarrow 1}(\hat{\mathbf{x}}); q^1(\mathsf{c}(\hat{\mathbf{x}})))$, i.e., condition (B13b) for $(m, 1)$.

We now show that (B13) holds for (m, s) when $s > 1$, given that (B13) holds for every pair $(\bar{m}, t)$ with $\bar{m} < m$ and $t \in S$ and every pair (m, t) with $t < s$. We argue first that $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) \subseteq \mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}})$. By our inductive assumption on $(m-1, s)$, it is sufficient to show that $\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m) \subseteq \mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}})$, as $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) = \mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^{m-1}) \cup \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m)$. Let $y \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m)$ be arbitrary. Since $y \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^m)$, under the choice procedure defining C^h, we must have that

$$y \in \bigcap_{t<s} R^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^m); q^t(\mathsf{c}(\mathbf{x}^m))). \tag{B17}$$

As each extended choice function C^t is substitutable, (B17) implies that

$$y \in \bigcap_{t<s} R^t(\mathsf{f}_{\rightsquigarrow t}(\mathbf{x}^m); q^t(\mathsf{c}(\mathbf{x}^m))). \tag{B18}$$

Now, by the inductive assumption (B13b) on pairs (m, t) for $t < s$, we have that $R^t(\mathsf{f}_{\rightsquigarrow t}(\mathbf{x}^m); q^t(\mathsf{c}(\mathbf{x}^m))) \subseteq R^t(\mathsf{f}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}})))$ for all $t < s$. Hence, (B18) implies that

$$y \in \bigcap_{t<s} R^t(\mathsf{f}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}}))). \tag{B19}$$

Applying condition (B1) of Claim 1 to $\hat{\mathbf{x}}$, we have that $C^t(\mathsf{c}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}}))) = C^t(\mathsf{f}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}})))$ for all t. Thus, if there were a $t < s$ such that $y \in C^t(\mathsf{c}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}})))$, we would have $y \in C^t(\mathsf{f}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}})))$, contradicting (B19). Hence, we must have

$$y \in \bigcap_{t<s} R^t(\mathsf{c}_{\rightsquigarrow t}(\hat{\mathbf{x}}); q^t(\mathsf{c}(\hat{\mathbf{x}}))).$$

Thus, there must exist some step $\hat{m}$ of the offer process $\hat{\mathbf{x}}$ such that $y \in \mathsf{c}_{\rightsquigarrow s}(\hat{\mathbf{x}}^{\hat{m}})$, and so $y \in \mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}}^{\hat{m}}) \subseteq \mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}})$; hence, we see that $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m) \subseteq \mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}})$, i.e., we have condition (B13a) for (m, s). Applying condition (B2) of Claim 1 then gives that $R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}^m); q^s(\mathsf{c}(\mathbf{x}^m))) \subseteq R^s(\mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}}); q^s(\mathsf{c}(\hat{\mathbf{x}})))$—condition (B13b) for (m, s). Having thus completed our induction, we have proven (B12).

Finally, we prove (B11), i.e., that $R^h(\mathsf{c}(\mathbf{x})) \subseteq R^h(\mathsf{c}(\hat{\mathbf{x}}))$. Suppose by way of contradiction that there exists some $y \in R^h(\mathsf{c}(\mathbf{x})) \smallsetminus R^h(\mathsf{c}(\hat{\mathbf{x}}))$. We have already shown (B12), that $\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}) \subseteq \mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}})$ for all $s \in S$; in particular,

$$y \in \mathsf{c}(\mathbf{x}) = \mathsf{f}_{\rightsquigarrow 1}(\mathbf{x}^m) \subseteq \mathsf{f}_{\rightsquigarrow 1}(\hat{\mathbf{x}}) = \mathsf{c}(\hat{\mathbf{x}}).$$

Thus, as $y \notin R^h(\mathsf{c}(\hat{\mathbf{x}}))$, we must have that $y \in C^h(\mathsf{c}(\hat{\mathbf{x}}))$. Therefore, there exists some division $s \in S$ such that $y \in C^s(\mathsf{c}_{\rightsquigarrow s}(\hat{\mathbf{x}}); q^s(\hat{\mathbf{x}}))$; by (B1) of Claim 1, we have that $y \in C^s(\mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}}); q^s(\hat{\mathbf{x}}))$ and so $y \notin R^s(\mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}}); q^s(\hat{\mathbf{x}}))$. By (B2) of Claim 1, since $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$, we then have that $y \notin R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathbf{x}))$. But as the extended choice function of s is substitutable, we must then have that $y \notin R^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathbf{x}))$—but this can only be the case if s or some division $t < s$ chooses y, which would contradict the assumption that $y \in R^h(\mathsf{c}(\mathbf{x}))$.

CLAIM 5: *The choice function C^h is not manipulable via contractual terms.*

PROOF:

By Proposition 5 of Hatfield et al. (2016), it is sufficient to show that when h is the only hospital, the following two conditions hold:

1) If $[\mathcal{C}(\succ_d, \succ_{D\smallsetminus\{d\}})]_d = \varnothing$, then either $[\mathcal{C}(\hat{\succ}_d, \succ_{D\smallsetminus\{d\}})]_d = \varnothing$ or $[\mathcal{C}(\hat{\succ}_d, \succ_{D\smallsetminus\{d\}})]_d = \{z^0\}$, and
2) if $[\mathcal{C}(\hat{\succ}_d, \succ_{D\smallsetminus\{d\}})]_d = \varnothing$, then $[\mathcal{C}(\succ_d, \succ_{D\smallsetminus\{d\}})]_d = \varnothing$.[3]

To show the first condition, we note that Claim 4 implies that if $[\mathcal{C}(\succ_d, \succ_{D\smallsetminus\{d\}})]_d = \varnothing$ and $z^0 \notin \mathcal{C}(\hat{\succ}_d, \succ_{D\smallsetminus\{d\}})$, then $R^h(\mathsf{c}(\mathbf{x})) \subseteq R^h(\mathsf{c}(\hat{\mathbf{x}}))$. Moreover, if $[\mathcal{C}(\succ_d, \succ_{D\smallsetminus\{d\}})]_d = \varnothing$, then $\{z^1, \ldots, z^N\} \subseteq R^h(\mathsf{c}(\mathbf{x}))$, and so combining the preceding two observations, we have that $\{z^1, \ldots, z^N\} \subseteq R^h(\mathsf{c}(\hat{\mathbf{x}}))$; hence $[\mathcal{C}(\hat{\succ}_d, \succ_{D\smallsetminus\{d\}})]_d = \varnothing$.

To show the second condition, note that by Proposition 1 of Hatfield et al. (2016), since the choice function of h is observably substitutable (Claim 2) and observably size monotonic (Claim 3), the cumulative offer mechanism outcome is order-independent. Thus we can consider $\mathbf{x}$ and $\hat{\mathbf{x}}$ to be generated by cumulative offer processes with respect to the proposal ordering $\vdash$ in which all of the contracts associated with doctors other than d precede all of the contracts associated with d, i.e., if $x \in X_d$ and $y \in X_{D\smallsetminus\{d\}}$, then $y \vdash x$.

Under our choice of $\vdash$, there must exist an $\bar{m}$ such that

1) $x^m = \hat{x}^m$ for all $m < \bar{m}$,
2) $x^{\bar{m}} = z^1$, and
3) $\hat{x}^{\bar{m}} = z^0$;

specifically, $\bar{m}$ is the first step of each cumulative offer process (with respect to $\vdash$) at which d proposes. Additionally, at each step after $\bar{m}$, exactly one contract is newly rejected and the offer process $\hat{\mathbf{x}}$ must end with the contract z^N being rejected, as

- z^N follows all contracts with doctors other than d under $\vdash$,
- the choice function of h is observably substitutable,
- the choice function of h is observably size monotonic, and
- we have assumed that $[\mathcal{C}(\hat{\succ}_d, \succ_{D\smallsetminus\{d\}})]_d = \varnothing$.

Thus, we must have that

1) $|R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{m}})) \smallsetminus R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{m}-1}))| = 1$ for all $\hat{m} \in \{\bar{m}, \bar{m}+1, \ldots, \hat{M}\}$, and

[3] Here, we use the notation introduced in (B9).

2) $z^N \in R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{M}})) \smallsetminus R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{M}-1}))$.

Similarly, for the offer process $\mathbf{x}$, we must have $|R^h(\mathsf{c}(\mathbf{x}^m)) \smallsetminus R^h(\mathsf{c}(\mathbf{x}^{m-1}))| = 1$ for all $m \in \{\bar{m}, \bar{m}+1, \ldots, M-1\}$.

Since $\mathbf{x}^{\bar{m}-1} = \hat{\mathbf{x}}^{\bar{m}-1}$, we have that $|C^h(\mathsf{c}(\mathbf{x}^{\bar{m}-1}))| = |C^h(\mathsf{c}(\hat{\mathbf{x}}^{\bar{m}-1}))|$. Moreover, since $|R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{m}})) \smallsetminus R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{m}-1}))| = 1$ for all $\hat{m} \in \{\bar{m}, \bar{m}+1, \ldots, \hat{M}\}$ (as we have just shown), we have that $|C^h(\mathsf{c}(\mathbf{x}^{\bar{m}-1}))| = |C^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{M}}))|$. Likewise, since $|R^h(\mathsf{c}(\mathbf{x}^m)) \smallsetminus R^h(\mathsf{c}(\mathbf{x}^{m-1}))| = 1$ for $m \in \{\bar{m}, \bar{m}+1, \ldots, M-1\}$ (as we have just shown), we have that $|C^h(\mathsf{c}(\mathbf{x}^{M-1}))| = |C^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{M}}))|$. But since $\mathsf{c}(\mathbf{x}^M) \subseteq \mathsf{c}(\hat{\mathbf{x}}^{\hat{M}})$, the observable size monotonicity of C^h implies that $|C^h(\mathsf{c}(\mathbf{x}^M))| \leq |C^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{M}}))|$. Thus, we must have

$$R^h(\mathsf{c}(\mathbf{x}^M)) \smallsetminus R^h(\mathsf{c}(\mathbf{x}^{M-1})) \neq \varnothing. \tag{B20}$$

Now, given (B20), suppose by way of contradiction that $y \in R^h(\mathsf{c}(\mathbf{x}^M)) \smallsetminus R^h(\mathsf{c}(\mathbf{x}^{M-1})) \neq \varnothing$ is not the contract z^N, and so is the least preferred acceptable contract with respect to $\succ_{\mathsf{d}(y)}$, where $\mathsf{d}(y) \neq d$. Claim 4 then implies there is some step $\hat{m} \geq \bar{m}$ such that $y \in R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{m}})) \smallsetminus R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{m}-1}))$. But, since $|R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{m}})) \smallsetminus R^h(\mathsf{c}(\hat{\mathbf{x}}^{\hat{m}-1}))| = 1$, and y is the least preferred acceptable contract for $\mathsf{d}(y)$, the cumulative offer process for $(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}})$ would end at $\hat{m}$ with the rejection of y, contradicting the fact that it ends with the rejection of z^N.

B5. Irrelevance of Rejected Contracts

Finally, we show that the choice function C^h satisfies the irrelevance of rejected contracts condition.

CLAIM 6: *The choice function C^h satisfies the irrelevance of rejected contracts condition.*

PROOF:

We suppose that z and Y are such that $z \in R^h(Y \cup \{z\})$. If $z \in R^h(Y \cup \{z\})$, then we must have

$$z \notin C^s([Y \cup \{z\}]_{\rightsquigarrow s}; q^s(Y \cup \{z\})) \tag{B21}$$

for each division s. Moreover, since each extended choice function C^s satisfies the irrelevance of rejected contracts condition, (B21) implies that

$$C^s([Y \cup \{z\}]_{\rightsquigarrow s}; q^s(Y \cup \{z\})) = C^s(Y_{\rightsquigarrow s}; q^s(Y \cup \{z\})) \tag{B22}$$

for all divisions s.

As the allotment function does not depend on irrelevant contracts, we have that $q(Y \cup \{z\}) = q(Y)$. Thus, we have

$$C^s([Y \cup \{z\}]_{\rightsquigarrow s}; q^s(Y \cup \{z\})) = C^s([Y \cup \{z\}]_{\rightsquigarrow s}; q^s(Y)). \tag{B23}$$

Combining (B22) and (B23) shows that

$$C^s([Y \cup \{z\}]_{\rightsquigarrow s}; q^s(Y \cup \{z\})) = C^s(Y_{\rightsquigarrow s}; q^s(Y))$$

for all divisions s; it follows that $C^h(Y \cup \{z\}) = C^h(Y)$.

C. A Multi-Division Choice Function with Flexible Allotments That Is Not Substitutably Completable

In many application contexts with non-substitutable preferences, stable and strategy-proof matching can be guaranteed by showing that each hospital's choice function is *substitutably* *completable* in the sense of Hatfield and Kominers (2016). However, our main result (Theorem 2) can not be demonstrated by using substitutable completability arguments; to prove this, we show in this appendix that there is a multi-division choice function with flexible allotments that is not substitutably completable.

Specifically, we give a multi-division choice function with flexible allotments that expresses the preferences introduced in Example 2 of Hatfield et al. (2016). First, we recall the setting: $H = \{h\}, D = \{d, e, f\}$, and $X = \{x, y, z, \hat{x}, \hat{z}\}$, with $\mathsf{h}(x) = \mathsf{h}(y) = \mathsf{h}(z) = \mathsf{h}(\hat{x}) = \mathsf{h}(\hat{z}) = h$, $\mathsf{d}(x) = \mathsf{d}(\hat{x}) = d$, $\mathsf{d}(y) = e$, and $\mathsf{d}(z) = \mathsf{d}(\hat{z}) = f$. We let the choice function C^h of h be induced by the preference relation[4]

$$\text{(C1)} \quad \{\hat{x}, z\} \succ \{x, \hat{z}\} \succ \{y, \hat{z}\} \succ \{\hat{x}, y\} \succ \{x, y\} \succ \{y, z\} \succ \{\hat{x}, \hat{z}\} \succ \{x, z\} \\ \succ \{y\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \varnothing.$$

Hatfield et al. (2016) proved that C^h does not have a substitutable completion. Here, we show that C^h can be modeled as a multi-division choice function with flexible allotments. We let $S = \{1, 2, 3, 4\}$, with the extended choice functions of the divisions s induced by the following preference relations:[5]

$$\begin{aligned}
&\succ_1 : \{\hat{x}\} \succ \emptyset, \\
&\succ_2 : \{x, y\} \succ \{x\} \succ \{y\} \succ \emptyset, \\
&\succ_3 : \{\hat{z}\} \succ \emptyset, \\
&\succ_4 : \{z\} \succ \emptyset.
\end{aligned}$$

It is immediate that for each $s \in S$, the extended choice function C^s is substitutable and size monotonic, satisfies the irrelevance of rejected contracts condition for any allotment, and, moreover, is monotonic with respect to the allotment and conditionally acceptant.

For each set of contracts $Y \subseteq X$, we denote the allotment function by

$$q(Y) = (q_1(Y), q_2(Y), q_3(Y), q_4(Y));$$

in Table C1, we define the allotment function for every possible set of contracts available to h, and also state the choice of h from that set of contracts.

It is clear from Table C1 that the choice function just defined is equivalent to the choice function induced by (C1). Moreover, it is straightforward to check using Table C1 that q does not depend on irrelevant contracts, does not observably grant excess positions, and is monotone in aggregate across observable offer processes.

We now show that q is single-peaked across observable offer processes. We say that a division s is *capacity constrained under* Y *at* n if $C^s(Y_{\rightsquigarrow s}; q^s(Y)) \subsetneq C^s(Y_{\rightsquigarrow s}; \infty)$ and $|C^s(Y_{\rightsquigarrow s}; q^s(Y))| = n$. Thus, to show that q is single-peaked across observable offer processes, it suffices to show that if a division is capacity constrained under $\mathsf{c}(\mathbf{x})$ at n for some offer process $\mathbf{x}$, for any offer process $\mathbf{y}$ such that $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\mathbf{y})$, division s is capacity constrained under $\mathsf{c}(\mathbf{y})$ at $m \leq n$. Now, we consider any observable offer processes $\mathbf{x}$ and $\mathbf{y}$ such that $\mathsf{c}(\mathbf{x}) \subsetneq \mathsf{c}(\mathbf{y})$. If $|\mathsf{c}(\mathbf{x})| \leq 2$, then no division is

[4]A preference relation $\succ_h$ for hospital h *induces* a choice function C^h for h under which

$$C^h(Y) = \max_{\succ_h}\{Z \subseteq X_h : Z \subseteq Y\},$$

where by $\max_{\succ_h}$ we mean the maximum with respect to the ordering $\succ_h$; that is, h chooses its most-preferred subset of Y.

[5]A preference relation $\succ_s$ for s *induces* an extended choice function C^s for s, under which

$$C^s(Y; a) = \max_{\succ_s}\{Z \subseteq Y : |Z| \leq a\}$$

where by $\max_{\succ_s}$ we mean the maximum with respect to the ordering $\succ_s$; that is, s chooses its most-preferred subset of Y that has size less than or equal to a.

Y	$q(Y)$	$C^h(Y)$
$\{x, \hat{x}, y, z, \hat{z}\}$	$(1, 0, 0, 1)$	$\{\hat{x}, z\}$
$\{x, \hat{x}, y, z\}$	$(1, 0, 0, 1)$	$\{\hat{x}, z\}$
$\{x, \hat{x}, z, \hat{z}\}$	$(1, 0, 0, 1)$	$\{\hat{x}, z\}$
$\{\hat{x}, y, z, \hat{z}\}$	$(1, 0, 0, 1)$	$\{\hat{x}, z\}$
$\{x, \hat{x}, z\}$	$(1, 0, 0, 1)$	$\{\hat{x}, z\}$
$\{\hat{x}, z, \hat{z}\}$	$(1, 0, 0, 1)$	$\{\hat{x}, z\}$
$\{\hat{x}, y, z\}$	$(1, 0, 0, 1)$	$\{\hat{x}, z\}$
$\{\hat{x}, z\}$	$(1, 0, 0, 1)$	$\{\hat{x}, z\}$
$\{x, \hat{x}, y, \hat{z}\}$	$(0, 1, 1, 0)$	$\{x, \hat{z}\}$
$\{x, y, z, \hat{z}\}$	$(0, 1, 1, 0)$	$\{x, \hat{z}\}$
$\{x, \hat{x}, \hat{z}\}$	$(0, 1, 1, 0)$	$\{x, \hat{z}\}$
$\{x, y, \hat{z}\}$	$(0, 1, 1, 0)$	$\{x, \hat{z}\}$
$\{x, z, \hat{z}\}$	$(0, 1, 1, 0)$	$\{x, \hat{z}\}$
$\{x, \hat{z}\}$	$(0, 1, 1, 0)$	$\{x, \hat{z}\}$
$\{\hat{x}, y, \hat{z}\}$	$(0, 1, 1, 0)$	$\{y, \hat{z}\}$
$\{y, z, \hat{z}\}$	$(0, 1, 1, 0)$	$\{y, \hat{z}\}$
$\{y, \hat{z}\}$	$(0, 1, 1, 0)$	$\{y, \hat{z}\}$
$\{x, \hat{x}, y\}$	$(1, 1, 0, 0)$	$\{\hat{x}, y\}$
$\{\hat{x}, y\}$	$(1, 1, 0, 0)$	$\{\hat{x}, y\}$
$\{x, y, z\}$	$(0, 2, 0, 0)$	$\{x, y\}$
$\{x, y\}$	$(0, 2, 0, 0)$	$\{x, y\}$
$\{y, z\}$	$(0, 1, 0, 1)$	$\{y, z\}$
$\{\hat{x}, \hat{z}\}$	$(1, 0, 1, 0)$	$\{\hat{x}, \hat{z}\}$
$\{x, z\}$	$(0, 1, 0, 1)$	$\{x, z\}$
$\{y\}$	$(0, 1, 0, 0)$	$\{y\}$
$\{z, \hat{z}\}$	$(0, 0, 1, 0)$	$\{\hat{z}\}$
$\{\hat{z}\}$	$(0, 0, 1, 0)$	$\{\hat{z}\}$
$\{x, \hat{x}\}$	$(1, 0, 0, 0)$	$\{\hat{x}\}$
$\{\hat{x}\}$	$(1, 0, 0, 0)$	$\{\hat{x}\}$
$\{x\}$	$(0, 1, 0, 0)$	$\{x\}$
$\{z\}$	$(0, 0, 0, 1)$	$\{z\}$
$\varnothing$	$(0, 0, 0, 0)$	$\varnothing$

Table C1—The value of the allotment function and choice function of h for every possible set of contracts available to h.

capacity constrained when $\mathsf{c}(\mathbf{x})$ is available, and so we have nothing to show. If $|\mathsf{c}(\mathbf{x})| = 3$, there are four cases to consider:

$\mathsf{c}(\mathbf{x}) = \{x, y, z\}$**.** In this case, division 4—and only division 4—is capacity constrained under $\mathsf{c}(\mathbf{x})$ at 0. Moreover, since $\mathbf{y}$ is observable and $\mathsf{c}(\mathbf{x}) \subsetneq \mathsf{c}(\mathbf{y})$, we must have that $\mathsf{c}(\mathbf{y}) = \{x, y, z, \hat{z}\}$, under which division 4 is still capacity constrained at 0.

$\mathsf{c}(\mathbf{x}) = \{\hat{x}, y, \hat{z}\}$**.** In this case, division 1—and only division 1—is capacity constrained under $\mathsf{c}(\mathbf{x})$ at 0. Moreover, since $\mathbf{y}$ is observable and $\mathsf{c}(\mathbf{x}) \subsetneq \mathsf{c}(\mathbf{y})$, we must have that $\mathsf{c}(\mathbf{y}) = \{\hat{x}, y, \hat{z}, x\}$, under which division 1 is still capacity constrained at 0.

$\mathsf{c}(\mathbf{x}) = \{x, y, \hat{z}\}$**.** In this case, $q(\mathsf{c}(\mathbf{x})) = (0, 1, 1, 0)$. Moreover, since $\mathbf{y}$ is observable and $\mathsf{c}(\mathbf{x}) \subsetneq \mathsf{c}(\mathbf{y})$, we must have that $\mathsf{c}(\mathbf{y}) = \{x, y, z, \hat{z}\}$; but then we also have $q(\mathsf{c}(\mathbf{y})) = (0, 1, 1, 0)$.

$\mathsf{c}(\mathbf{x}) = \{\hat{x}, y, z\}$**.** In this case, there does not exist an observable offer process $\mathbf{y}$ such that $\mathsf{c}(\mathbf{x}) \subsetneq \mathsf{c}(\mathbf{y})$.

D. Matching with Distributional Constraints

In this appendix, we explain how our framework nests the matching with regional caps model of Kamada and Kojima (2015, 2017).

D1. The Kamada and Kojima (2015, 2017) Model of Matching with Distributional Constraints

First, we must note one small, potentially confusing point of terminology: In the Kamada and Kojima (2015, 2017) framework, hospitals are partitioned into regions, and there are distributional constraints (regional caps) that restrict the number of doctors that can be assigned to each region. In our framework, by contrast, hospitals are the top-level institutions, and constraints within hospitals determine distributions across divisions. Thus, the divisions in our framework correspond to Kamada and Kojima's (2015, 2017) *hospitals*, while our hospitals correspond to Kamada and Kojima's (2015, 2017) *regions*.

In the setting of Kamada and Kojima (2015, 2017), doctors have strict preferences over the jobs they could have—represented in our context by contracts

$$(d, h, s) \in X = \bigcup_{h \in H} D \times \{h\} \times S^h$$

or, equivalently, by hospital–division pairs

$$(h, s) \in J \equiv \bigcup_{h \in H} \{(h, s) : s \in S^h\}.^{6,7}$$

The preferences of doctor d over contracts in X_d naturally correspond to preferences over hospital–division pairs: If $(d, h, s) \succ_d (d, g, t)$ for some hospitals $h, g \in H$, some division $s \in S^h$, and some division $t \in S^g$, then we write $(h, s) \succ_d (g, t)$. Similarly, if $(d, h, s) \succ_d \emptyset$ for some hospital $h \in H$ and some division $s \in S^h$, then we write $(h, s) \succ_d \emptyset$. Finally, if $\emptyset \succ_d (d, h, s)$ for some hospital $h \in H$ and some division $s \in S^h$, then we write $\emptyset \succ_d (h, s)$.

Each hospital–division pair $j \in J$ has *responsive preferences* with respect to some strict ranking $\succ_j$ of $D \cup \{\emptyset\}$ and a fixed *capacity* $\bar{q}^j$. That is, from any given set of contracts Y, the choice

[6] In the Kamada and Kojima (2015, 2017) model, doctors' preferences could be expressed as rankings over just divisions, as Kamada and Kojima (2015, 2017) took divisions as primitives. In our context, doctors' preferences must be expressed over hospital–division pairs (h, s), as divisions in our setup only exist when associated with a hospitals.

[7] Here, unlike in the main text, we explicitly note the dependence of the set of divisions S on the underlying hospital, as we need to consider doctors' preferences over hospital–division pairs.

function C^j of the hospital–division pair $j = (h, s)$ chooses the contracts in

$$Y_j \equiv \left\{(d, \hat{h}, \hat{s}) \in Y : j = (\hat{h}, \hat{s}) = (h, s)\right\}$$

associated to the $\bar{q}^j$ most highly-ranked doctors according to $\succ_j$; if there are fewer than $\bar{q}^j$ contracts in Y_j associated with doctors that j ranks more highly than the outside option $\emptyset$, then choice function C^j of j chooses all such contracts. Additionally, each hospital h has an *overall capacity* $\bar{Q}^h$.

A *matching* is a mapping μ that assigns doctors to hospital–division pairs, i.e., a mapping μ such that

1) $\mu(d) \in J \cup \{\emptyset\}$ for all $d \in D$,
2) $\mu(j) \subseteq D$ for all $j \in H$, and
3) for all $d \in D$ and $h \in H$, we have that $d \in \mu(j)$ if and only if $\mu(d) = j$.

A matching μ is

- *individually rational for doctors* if, for all $d \in D$, we have $\mu(d) \succ_d \emptyset$;
- *individually rational for hospital divisions* if, for all hospital–division pairs $j \in J$, we have $d \succ_j \emptyset$ whenever $d \in \mu(j)$;
- *feasible* if $|\mu(j)| \leq \bar{q}^j$ for each $j \in J$ and $\sum_{s \in S^h} |\mu((h, s))| \leq \bar{Q}^h$ for each $h \in H$; and
- *blocked by* $(d, j) \in D \times J$ if $j \succ_d \mu(d)$, $d \succ_j \emptyset$, and either $|\mu(j)| < \bar{q}^j$ or $d \succ_j e$ for some $e \in \mu(j)$.

At the aggregate level, the hospital h has preferences over capacity distributions across divisions. Formally, there is a weak ordering $\trianglerighteq_h$ over the set of *distribution vectors*

$$\mathcal{W}^h \equiv \{(w_s)_{s \in S^h} : w_s \in \mathbb{Z}_{\geq 0}\}.$$

Given $\trianglerighteq_h$, a *capacity allocation rule* is a mapping $p : \mathcal{W}^h \to \mathcal{W}^h$ such that for all $w \in \mathcal{W}^h$,

$$p(w) \in \max_{\trianglerighteq_h}\{w' : w' \leq w\}.^{8,9}$$

Kamada and Kojima (2015, 2017) imposed the following assumptions on the capacity allocation rule p:

1) If $w, w' \in \mathcal{W}^h$ are such that $p(w) \leq w' \leq w$, then $p(w') = p(w)$.
2) For all $w \in \mathcal{W}^h$ and all $s \in S^h$, $[p(w)]_s \leq \bar{q}^{(h,s)}$.
3) For all $w \in \mathcal{W}^h$, $\sum_{s \in S^h}[p(w)]_s \leq \bar{Q}^h$.
4) For all $w \in \mathcal{W}^h$, if there is an $s \in S^h$ such that $[p(w)]_s < \min\{w_s, \bar{q}^{(h,s)}\}$, then $\sum_{t \in S^h}[p(w)]_t = \bar{Q}^h$.
5) For all $w, w' \in \mathcal{W}^h$ and $s \in S^h$ such that $w \leq w'$ and $[p(w)]_s < [p(w')]_s$, we have $[p(w)]_s = w_s$.

In the sequel, we assume the preceding conditions, and refer to them as *Conditions 1–5 of Kamada and Kojima (2015, 2017)*.

Kamada and Kojima (2015, 2017) also introduced the following stability concept, *stability under distributional constraints*.

[8] Here, we suppress the dependence of p on the hospital h for notational simplicity.
[9] Here, by $\max_{\trianglerighteq_h}$, we mean the maximum with respect to the ordering $\trianglerighteq_h$.

DEFINITION 1: *A matching* μ *is* stable under distributional constraints*, if it is feasible and individually rational for doctors and hospitals, and whenever* $(d,(h,s))$ *is a blocking pair the following three conditions hold:*

1) *The hospital* h *is capacity constrained, i.e.,* $\sum_{\hat{s}\in S^h}|\mu((h,\hat{s}))|=\bar{Q}^h$.

2) *The hospital–division pair* (h,s) *prefers all of its doctors under* μ *to* d*, i.e.,* $d' \succ_{(h,s)} d$ *for all* $d' \in \mu((h,s))$.

3) *Either*

   a) *doctor* d *is not employed at hospital* h *under* μ*, i.e.,* $d \notin \cup_{s\in S^h}\mu((h,s))$*, or*

   b) *hospital* h *prefers its distribution vector under* μ *to the one that would arise if* d *were to switch to* j*, that is,* $(|\mu((h,\hat{s}))|)_{\hat{s}\in S^h} \trianglerighteq_h v$*, where*

$$v_{\hat{s}}=\begin{cases}|\mu((h,\hat{s}))|+1 & \hat{s}=s\\ |\mu((h,\hat{s}))|-1 & (h,\hat{s})=\mu(d)\\ |\mu((h,\hat{s}))| & \textit{otherwise.}\end{cases}$$

Definition 1 rules out blocks $(d,(h,s))$ in which the hospital h is capacity constrained, the division s only benefits if it adds d as a new doctor, and either d is employed at a different hospital pre-block or the hospital h prefers its distribution vector pre-block to its distribution vector post-block.

D2. Embedding the Kamada–Kojima (2015, 2017) Model within Our Framework

We now show how to embed the model of Kamada and Kojima (2015, 2017) into our model of matching with flexible allotments. For notational simplicity, we focus on a single hospital h, and return to suppressing the notation for h wherever doing so will not introduce confusion. Now, for each doctor d and division $s\in S^h$, there is just one contract, denoted $(d,s)=(d,h,s)$, under which d is employed at division s (of h). Thus, the set of contracts can be reduced to $X=\cup_{s\in S^h}\{(d,s):d\in D\}$.

We assume that each division $s\in S^h$ has a strict ranking $\succ_s$ of contracts in the set

$$\{(d,s)\in X: d\in D\}\cup\{\emptyset\}$$

such that

1) $(d,s)\succ_s (d',s)$ if and only if $d\succ_{(h,s)} d'$,

2) $(d,s)\succ_s \emptyset$ if and only if $d\succ_{(h,s)}\emptyset$, and

3) $\emptyset\succ_s (d,s')$ for all $s'\neq s$ and $d\in D$.[10]

For any allotment a, the extended choice function $C^s(\cdot;a)$ of each division s is assumed to be responsive with respect to the order $\succ_s$ with capacity a.[11] Next, we use p to define an allotment function q by setting, for each $Y\subseteq X$,

$$q(Y)=p(\varpi(Y)), \tag{D1}$$

[10]Note that different divisions never have acceptable contracts in common.

[11]We could instead use the capacity $\min\{a,\bar{q}^{(h,s)}\}$, which bounds the number of accepted doctors at $\bar{q}^{(h,s)}$. This is not formally necessary in our construction, however, as we impose the division cap $\bar{q}^{(h,s)}$ via the allotment function.

where we take

$$\varpi(Y) \equiv (|\{y \in Y_h : y \succ_{(h,s)} \emptyset\}|)_{s \in S} \tag{D2}$$

to be the vector that counts the number of contracts in Y_h that are acceptable to each division s.

With our specifications of the extended choice functions C^s and the allotment function q, the choice function C^h implements the choices made by the hospital–division pairs (h, s) under the choice functions $C^{(h,s)}$ and the constraints imposed by p under the cumulative offer mechanism.

We now prove that the extended choice functions C^s, combined with the allotment function q, do indeed make the induced hospital choice function C^h a multi-division choice function with flexible allotments.

CLAIM 7: *The hospital choice function C^h induced by $(C^s)_{s \in S}$ and q (under the choice procedure defined in Section III) is a multi-division choice function with flexible allotments. That is:*

- *For any fixed allotment, each division's extended choice function $C^s(\cdot; a)$ is substitutable and size monotonic, and satisfies the irrelevance of rejected contracts condition. Moreover, each division's extended choice function C^s is monotonic with respect to the allotment and conditionally acceptant.*
- *The allotment function q does not depend on irrelevant contracts, does not observably grant excess positions, is single-peaked across observable offer processes, and is monotone in aggregate across observable offer processes.*

PROOF:

As $C^s(\cdot; a)$ is responsive, it is immediate that it satisfies the classical substitutability, size monotonicity, and irrelevance of rejected contracts conditions. Likewise, it is immediate that each division's extended choice function C^s is monotonic with respect to the allotment and conditionally acceptant.

Now, we show the claimed properties of the allotment function q; as we do so, we sometimes abuse notation slightly by writing

$$p_s(w) \equiv (p(w))_s \quad \text{and} \quad \varpi_s(Y) \equiv (\varpi(Y))_s.$$

1) **The allotment function q does not depend on irrelevant contracts:** We consider a set of contracts $Y \subseteq X$ and suppose that $z \in R^h(Y)$. We first argue that

$$q(Y) = p(\varpi(Y)) \le \varpi(Y \smallsetminus \{z\}). \tag{D3}$$

First, we recall that the contract z is associated to a unique division—that is, $z = (d, s)$ for some division s. Thus, if $z = (d, s) \in R^h(Y)$, we must have either:

- $\emptyset \succ_{(h,s)} d$ (d is unacceptable to s),
- $d \succ_{(h,s)} \emptyset$ and $p_s(\varpi(Y)) = \bar{q}^{(h,s)}$ (d is acceptable to s, but s is at maximum-possible capacity), or
- $d \succ_{(h,s)} \emptyset$ and $p_s(\varpi(Y)) < \bar{q}^{(h,s)}$ (d is acceptable to s, but the allotment rule constrains s below its maximum-possible capacity).

In the first case, we must have $\varpi_t(Y) = \varpi_t(Y \smallsetminus \{z\})$ for all $t \in S$ (recall (D2)); hence, (D3) is immediate, as $p_t(\varpi_t(Y)) \le \varpi_t(Y)$ by construction.

In the second and third cases, we have that $q^s(Y) \le \varpi_s(Y) - 1$ given that $z \in R^h(Y)$ and $d \succ_{(h,s)} \emptyset$; as $p_s(\varpi(Y)) = q^s(Y)$, this implies that

$$p_s(\varpi(Y)) \le \varpi_s(Y) - 1. \tag{D4}$$

Meanwhile, we have

$$\varpi_s(Y \smallsetminus \{z\}) = \varpi_s(Y) - 1, \tag{D5}$$

as $|\{y \in Y_h \smallsetminus \{z\} : y \succ_{(h,s)} \emptyset| = |\{y \in Y_h : y \succ_{(h,s)} \emptyset| - 1$. Combining (D4) with (D5) shows that

$$p_s(\varpi(Y)) \leq \varpi_s(Y \smallsetminus \{z\}). \tag{D6}$$

Meanwhile, as z is associated to s, it is unacceptable to all divisions $t \neq s$; this implies that $\varpi_t(Y \smallsetminus \{z\}) = \varpi_t(Y)$ for all such t. As $p_t(\varpi(Y)) \leq \varpi_t(Y)$ for all $t \in S$, we then have

$$p_t(\varpi(Y)) \leq \varpi_t(Y \smallsetminus \{z\}) \quad \text{for all divisions } t \neq s. \tag{D7}$$

Combining (D6) with (D7), we find that

$$p_t(\varpi(Y)) \leq \varpi_t(Y \smallsetminus \{z\})$$

for *all* divisions $t \in S$—exactly (D3).

Now, having proven (D3), we note that $\varpi_t(Y \smallsetminus \{z\}) \leq \varpi_t(Y)$ mechanically, so that Condition 1 of Kamada and Kojima (2015, 2017) implies that $p(\varpi(Y)) = p(\varpi(Y \smallsetminus \{z\}))$. Thus, we have $q(Y \smallsetminus \{z\}) = p(\varpi(Y \smallsetminus \{z\})) = p(\varpi(Y)) = q(Y)$; this implies the claim.

2) **The allotment function q does not observably grant excess positions:** Let $\mathbf{x} = (x^1, \ldots, x^M)$ be an observable offer process, and assume that there is a division s such that

$$q^s(\mathsf{c}(\mathbf{x})) > \min\{|\{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}) : d \succ_{(h,s)} \emptyset\}|, \bar{q}^{(h,s)}\}. \tag{D8}$$

We assume without loss of generality that[12]

$$\begin{aligned} &\text{for all pairs } (m,t) \text{ such that either } m < M \text{ or } m = M \text{ and } t < s, \\ &\text{we have } q^t(\mathsf{c}(\mathbf{x}^m)) \leq \min\{|\{(d,t) \in \mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^m) : d \succ_{(h,t)} \emptyset\}|, \bar{q}^{(h,t)}\}. \end{aligned} \tag{D9}$$

Now, as $|\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\mathbf{x}^{M-1})| = 1$, we must have $p_t(\varpi(\mathsf{c}(\mathbf{x}))) \leq p_t(\varpi(\mathsf{c}(\mathbf{x}^{M-1}))) + 1$ for all t. Moreover, by Condition 5 of Kamada and Kojima (2015, 2017), we have $p_t(\varpi(\mathsf{c}(\mathbf{x}))) = p_t(\varpi(\mathsf{c}(\mathbf{x}^{M-1}))) + 1$ only if $p_t(\varpi(\mathsf{c}(\mathbf{x}^{M-1}))) = \varpi_t(\mathsf{c}(\mathbf{x}^{M-1}))$.

Combining the preceding observations with (D9) (and recalling that $q(\cdot) = p(\varpi(\cdot))$ by (D1)), we see that for all pairs (m,t) such that either $m < M$ or $m = M$ and $t < s$, we must have either

$$q^t(\mathsf{c}(\mathbf{x})) \leq q^t(\mathsf{c}(\mathbf{x}^{M-1})) \tag{D10}$$

or

$$\begin{gathered} q^t(\mathsf{c}(\mathbf{x})) = q^t(\mathsf{c}(\mathbf{x}^{M-1})) + 1 \\ q^t(\mathsf{c}(\mathbf{x}^{M-1})) = |\{(d,t) \in \mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{M-1}) : d \succ_{(h,t)} \emptyset\}| < \bar{q}^{(h,t)} \\ \mathsf{d}(x^M) \succ_{(h,t)} \emptyset \\ x^M \in \mathsf{c}_{\rightsquigarrow t}(\mathbf{x}) \smallsetminus \mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{M-1}). \end{gathered} \tag{D11}$$

[12] If (D9) did not hold for the claimed pairs (m,t), then we could shorten the offer process $\mathbf{x}$ to $\mathbf{x}^m$ or replace s with t.

Now, we prove via induction that

$$C^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{M-1}); q^t(\mathsf{c}(\mathbf{x}^{M-1}))) \subseteq \mathsf{c}_{\rightsquigarrow t}(\mathbf{x}) \text{ for all } t \leq s. \tag{D12}$$

First, we note that (D12) is immediately satisfied for the base case of $t = 1$, as

$$\mathsf{c}_{\rightsquigarrow 1}(\mathbf{x}^{M-1}) = \mathsf{c}(\mathbf{x}^{M-1}) \subseteq \mathsf{c}(\mathbf{x}) = \mathsf{c}_{\rightsquigarrow 1}(\mathbf{x}).$$

We suppose that (D12) holds for all $t' < t \leq s$, and fix some $y \in C^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{M-1}); q^t(\mathsf{c}(\mathbf{x}^{M-1})))$. Since $y \in C^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{M-1}); q^t(\mathsf{c}(\mathbf{x}^{M-1})))$, under the choice procedure defining C^h, for any $t' < t$, we must have that the contract y is not among the $q^{t'}(\mathsf{c}(\mathbf{x}^{M-1}))$ most preferred contracts in $\{(d,t') \in \mathsf{c}_{\rightsquigarrow t'}(\mathbf{x}^{M-1}) : d \succ_{(h,t')} \emptyset\}$.

If $q^{t'}(\mathsf{c}(\mathbf{x})) \leq q^{t'}(\mathsf{c}(\mathbf{x}^{M-1}))$, then (D12) in the t' case immediately implies that y is not among the $q^{t'}(\mathsf{c}(\mathbf{x}))$ most preferred contracts in $\{(d,t') \in \mathsf{c}_{\rightsquigarrow t'}(\mathbf{x}) : d \succ_{(h,t')} \emptyset\}$.

Otherwise, if $q^{t'}(\mathsf{c}(\mathbf{x})) = q^{t'}(\mathsf{c}(\mathbf{x}^{M-1})) + 1$, we must have that

$$q^{t'}(\mathsf{c}(\mathbf{x}^{M-1})) = |\{(d,t') \in \mathsf{c}_{\rightsquigarrow t'}(\mathbf{x}^{M-1}) : d \succ_{(h,t')} \emptyset\}| < \bar{q}^{(h,t')},$$

recalling (D11). As y is not chosen by t' when $\mathsf{c}(\mathbf{x}^{M-1})$ is available to h, the preceding observation implies that either $\emptyset \succ_{(h,t')} \mathsf{d}(y)$ or $(\mathsf{d}(y), t') \notin \mathsf{c}(\mathbf{x}^{M-1})$. In the former case, it is immediate that $\mathsf{d}(y) \notin \mathsf{d}(C^{t'}(\mathsf{c}_{\rightsquigarrow t'}(\mathbf{x}); q^{t'}(\mathsf{c}(\mathbf{x}))))$. In the latter case, $y \in C^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{M-1}); q^t(\mathsf{c}(\mathbf{x}^{M-1})))$ implies $\mathsf{d}(y) \neq \mathsf{d}(x^M)$ and thus $(\mathsf{d}(y), t') \notin \mathsf{c}(\mathbf{x})$.

Hence, no matter whether $q^{t'}(\mathsf{c}(\mathbf{x})) \leq q^{t'}(\mathsf{c}(\mathbf{x}^{M-1}))$ or $q^{t'}(\mathsf{c}(\mathbf{x})) = q^{t'}(\mathsf{c}(\mathbf{x}^{M-1})) + 1$, we obtain that $\mathsf{d}(y) \notin \mathsf{d}(C^{t'}(\mathsf{c}_{\rightsquigarrow t'}(\mathbf{x}); q^{t'}(\mathsf{c}(\mathbf{x}))))$. Since y was arbitrary, this shows that $C^t(\mathsf{c}_{\rightsquigarrow t}(\mathbf{x}^{M-1}); q^t(\mathsf{c}(\mathbf{x}^{M-1}))) \subseteq \mathsf{c}_{\rightsquigarrow t}(\mathbf{x})$ if $C^{t'}(\mathsf{c}_{\rightsquigarrow t'}(\mathbf{x}^{M-1}); q^{t'}(\mathsf{c}(\mathbf{x}^{M-1}))) \subseteq \mathsf{c}_{\rightsquigarrow t'}(\mathbf{x})$ for all $t' < t$; this completes the proof of (D12).

To complete the proof that q does not observably grant excess positions, we now derive a contradiction to (D8). Note again from (D9) that

$$q^s(\mathsf{c}(\mathbf{x}^{M-1})) \leq \min\{|\{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}) : d \succ_{(h,s)} \emptyset\}|, \bar{q}^{(h,s)}\};$$

this inequality implies that $|C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}); q^s(\mathsf{c}(\mathbf{x}^{M-1})))| = q^s(\mathsf{c}(\mathbf{x}^{M-1}))$. There are two cases to consider:

- We consider first the case in which $q^s(\mathsf{c}(\mathbf{x})) \leq q^s(\mathsf{c}(\mathbf{x}^{M-1}))$. Given (D12), we have $C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}); q^s(\mathsf{c}(\mathbf{x}^{M-1}))) \subseteq \mathsf{c}_{\rightsquigarrow s}(\mathbf{x})$, and thus

$$|C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}); q^s(\mathsf{c}(\mathbf{x}^{M-1})))| \leq |\{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}) : d \succ_{(h,s)} \emptyset\}|.$$

  Since $|C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}); q^s(\mathsf{c}(\mathbf{x}^{M-1})))| = q^s(\mathsf{c}(\mathbf{x}^{M-1}))$ and $q^s(\mathsf{c}(\mathbf{x})) \leq q^s(\mathsf{c}(\mathbf{x}^{M-1}))$, we obtain a contradiction to (D8).
- We consider second the case in which $q^s(\mathsf{c}(\mathbf{x})) > q^s(\mathsf{c}(\mathbf{x}^{M-1}))$. Note that, here again, (D11) must hold. Since we then have $q^s(\mathsf{c}(\mathbf{x}^{M-1})) = |\{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}) : d \succ_{(h,s)} \emptyset\}|$ and $|\{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}) : d \succ_{(h,s)} \emptyset\}| < \bar{q}^{(h,s)}$, we see that $C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}); q^s(\mathsf{c}(\mathbf{x}^{M-1}))) = \{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}) : d \succ_{(h,s)} \emptyset\}$. Since $C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}); q^s(\mathsf{c}(\mathbf{x}^{M-1}))) \subseteq \mathsf{c}_{\rightsquigarrow s}(\mathbf{x})$ and $x^M \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}) \smallsetminus C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}); q^s(\mathsf{c}(\mathbf{x}^{M-1})))$, we have that

$$|\{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}) : d \succ_{(h,s)} \emptyset\}| = |\{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}) : d \succ_{(h,s)} \emptyset\}| + 1.$$

  Since, again by (D11), we have that $|\{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}^{M-1}) : d \succ_{(h,s)} \emptyset\}| = q^s(\mathsf{c}(\mathbf{x}^{M-1}))$

and $q^s(\mathsf{c}(\mathbf{x})) = q^s(\mathsf{c}(\mathbf{x}^{M-1}))+1$, we obtain that $|\{(d,s) \in \mathsf{c}_{\rightsquigarrow s}(\mathbf{x}) : d \succ_{(h,s)} \emptyset\}| = q^s(\mathsf{c}(\mathbf{x}))$, again contradicting our earlier assumption (D8).

3) **The allotment function q is single-peaked across observable offer processes:** We consider observable offer processes $\mathbf{x}$ and $\mathbf{y}$ such that $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\mathbf{y})$. Let s be a division such that $|C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); \infty)| > q^s(\mathsf{c}(\mathbf{x}))$. As $\varpi_s(\mathsf{c}(\mathbf{x})) \geq |C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); \infty)|$ automatically, we see that $\varpi_s(\mathsf{c}(\mathbf{x})) > q^s(\mathsf{c}(\mathbf{x}))$. As $q(\mathsf{c}(\mathbf{x})) = p(\varpi(\mathsf{c}(\mathbf{x})))$ by construction (recall (D1)) and $\varpi(\mathsf{c}(\mathbf{y})) \geq \varpi(\mathsf{c}(\mathbf{x}))$ automatically, Condition 5 of Kamada and Kojima (2015, 2017) then implies that $p_s(\varpi(\mathsf{c}(\mathbf{y}))) \leq p_s(\varpi(\mathsf{c}(\mathbf{x})))$, so that $q^s(\mathsf{c}(\mathbf{y})) \leq q^s(\mathsf{c}(\mathbf{x}))$, as desired.

4) **The allotment function q is monotone in aggregate across observable offer processes:** We consider observable offer processes $\mathbf{x}$ and $\mathbf{y}$ such that $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\mathbf{y})$. If we have

$$\sum_{s\in S} q^s(\mathsf{c}(\mathbf{x})) > \sum_{s\in S} q^s(\mathsf{c}(\mathbf{y})), \tag{D13}$$

then it must be the case that $\sum_{s\in S} q^s(\mathsf{c}(\mathbf{y})) < \bar{Q}^h$, and so, by Condition 4 of Kamada and Kojima (2015, 2017), we have that $q^s(\mathsf{c}(\mathbf{y})) = p_s(\varpi(\mathsf{c}(\mathbf{y}))) = \min\{\varpi_s(\mathsf{c}(\mathbf{y})), \bar{q}^{(h,s)}\}$ for all s (as, by the definition of the capacity allocation rule, $q^s(\mathsf{c}(\mathbf{y})) \leq \varpi_s(\mathsf{c}(\mathbf{y}))$, and, by Condition 2 of Kamada and Kojima (2015, 2017), $q^s(\mathsf{c}(\mathbf{y})) \leq \bar{q}^{(h,s)}$). But, since $\mathsf{c}(\mathbf{x}) \subseteq \mathsf{c}(\mathbf{y})$, we have that $\min\{\varpi_s(\mathsf{c}(\mathbf{x})), \bar{q}^{(h,s)}\} \leq \min\{\varpi_s(\mathsf{c}(\mathbf{y})), \bar{q}^{(h,s)}\}$ for all s. Thus, by Condition 4 of Kamada and Kojima (2015, 2017),

$$q^s(\mathsf{c}(\mathbf{x})) = p_s(\varpi(\mathsf{c}(\mathbf{x}))) \leq \min\{\varpi_s(\mathsf{c}(\mathbf{x})), \bar{q}^{(h,s)}\} \leq \min\{\varpi_s(\mathsf{c}(\mathbf{y})), \bar{q}^{(h,s)}\} = q^s(\mathsf{c}(\mathbf{y})).$$

But then, summing over all s, we have that

$$\sum_{s\in S} q^s(\mathsf{c}(\mathbf{x})) \leq \sum_{s\in S} q^s(\mathsf{c}(\mathbf{y})),$$

contradicting (D13).

D3. Strategy-Proofness of Cumulative Offer Mechanisms in the Kamada–Kojima (2015, 2017) Setting

With the embedding described in Appendix D2, strategy-proofness of cumulative offer mechanisms in the Kamada and Kojima (2015, 2017) context follows directly from our main results.

PROPOSITION 1: *Suppose that each hospital has a choice function constructed as in the model of Kamada and Kojima (2015, 2017) (as described in Appendix D1). Then any cumulative offer mechanism is strategy-proof.*

PROOF:
This follows immediately upon combining Claim 7 with Corollary 1.

D4. Stability of Cumulative Offer Mechanism Outcomes Under Distributional Constraints

Finally, we show that cumulative offer mechanism outcomes in our context correspond to matchings that are stable under distributional constraints.

For each feasible set of contracts $Y \subseteq X$, we define a matching μ^Y by setting

$$\mu^Y(d) = \begin{cases} (h,s) & (d,h,s) \in Y \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\mu^Y((h,s)) = \{d : (d,h,s) \in Y\}.$$

PROPOSITION 2: *Suppose that each hospital has a choice function constructed as in the model of Kamada and Kojima (2015, 2017) (as described in Appendix D1). Then if A is the outcome of a cumulative offer mechanism, the matching μ^A is stable under distributional constraints.*

PROOF:

We fix the preferences of the doctors and hospital–division pairs as well as the capacity allocation rules of hospitals, and use the construction described in Appendix D2 to formulate the associated choice functions and preferences over contracts, as well as the allotment functions. We let $\mathbf{x} = (x^1, \ldots, x^M)$ be the sequence of contracts proposed under any cumulative offer mechanism given those preferences.[13] With this setup, we have

$$A = \bigcup_{h \in H} C^h(\mathsf{c}(\mathbf{x})).$$

It is immediate that μ^A is

- feasible and
- individually rational for both doctors and hospital divisions.

Now, suppose that μ^A is blocked by $(d, (h, s))$, i.e., $(h, s) \succ_d \mu^A(d)$ and $d \succ_{(h,s)} \emptyset$.

We argue first that $\hat{d} \succ_{(h,s)} d$ for all $\hat{d} \in \mu^A(s)$ (i.e., Condition 2 of Definition 1). Since $(h, s) \succ_d \mu^A(d)$, we must have that $x = (d, h, s)$ was proposed at some step of the cumulative offer process corresponding to $\mathbf{x}$. Since $(d, h, s) \notin A$, there has to exist an $\hat{M}$ such that, letting $\hat{\mathbf{x}} \equiv (x^1, \ldots, x^{\hat{M}})$, we have that $(d, h, s) \in R^s(\mathsf{c}_{\rightsquigarrow s}(\hat{\mathbf{x}}); q^s(\mathsf{c}(\hat{\mathbf{x}})))$. By (B1) of Claim 1, we must have that $(d, h, s) \in R^s(\mathsf{f}_{\rightsquigarrow s}(\hat{\mathbf{x}}); q^s(\mathsf{c}(\hat{\mathbf{x}})))$. By (B2) of Claim 1, we must then have that

$$(d, h, s) \in R^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x}))). \tag{D14}$$

Now, if $\hat{d} \in \mu^A((h, s))$, then $(\hat{d}, h, s) \in C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x})))$. Moreover, by (B1) of Claim 1, we must have that $C^s(\mathsf{c}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x}))) = C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x})))$, and so

$$(\hat{d}, h, s) \in C^s(\mathsf{f}_{\rightsquigarrow s}(\mathbf{x}); q^s(\mathsf{c}(\mathbf{x}))). \tag{D15}$$

Together, (D14) and (D15) imply that $\hat{d} \succ_{(h,s)} d$ for all $\hat{d} \in \mu^A(s)$, as desired.

We argue second that h is capacity constrained (Condition 1 of Definition 1). We have just shown that $\hat{d} \succ_{(h,s)} d$ for all $\hat{d} \in \mu^A(s)$. Nevertheless, $(d, (h, s))$ blocks μ; hence, it must be the case that $|\mu^A((h, s))| < \bar{q}^{(h,s)}$.[14] Moreover, as $\mu(d) \neq (h, s)$ under μ even though (d, h, s) is both

- in $\mathsf{c}(\mathbf{x})$ and
- acceptable to (h, s),

we must have $|\mu^A((h, s))| < \varpi_s(\mathsf{c}(\mathbf{x}))$. Thus, we have

$$p_s(\varpi(\mathsf{c}(\mathbf{x}))) = q^s(\mathsf{c}(\mathbf{x})) = |\mu^A((h, s))| < \min\{\varpi_s(\mathsf{c}(\mathbf{x})), \bar{q}^{(h,s)}\};$$

hence, Condition 4 of Kamada and Kojima (2015, 2017) implies that hospital h is capacity constrained, as desired.

[13] By Proposition 1 of Hatfield et al. (2016), all cumulative offer mechanisms in this context are outcome-equivalent.

[14] If we had $|\mu^A((h, s))| = \bar{q}^{(h,s)}$ then $(d, (h, s))$ could not block μ, as under μ the hospital–division pair (h, s) would be assigned $\bar{q}^{(h,s)}$ doctors it prefers to d.

Finally, we show Condition 3 of Definition 1. Suppose that d is employed at h under μ (Case 3b of Condition 3), i.e., there exists a division $\hat{s} \in S^h$ such that $\mu(d) = (h, \hat{s})$. By construction, we have

$$p(\varpi(\mathsf{c}(\mathbf{x}))) = q(\mathsf{c}(\mathbf{x})) = (|\mu^A((h,t))|)_{t \in S^h}.$$

Let v be obtained by setting $v_s = p_s(\varpi(\mathsf{c}(\mathbf{x}))) + 1$, $v_{\hat{s}} = p_{\hat{s}}(\varpi(\mathsf{c}(\mathbf{x}))) - 1$, and $v_t = p_t(\varpi(\mathsf{c}(\mathbf{x})))$ for all $t \in S^h \smallsetminus \{s, \hat{s}\}$. As $(d,h,s) \in \mathsf{c}(\mathbf{x})$ and $(d,h,s) \notin \mu^A((h,s))$, we must have $v_s < \varpi_s(\mathsf{c}(\mathbf{x}))$, so that $v \leq \varpi(\mathsf{c}(\mathbf{x}))$. But then, $v \in \{w' : w' \leq \varpi(\mathsf{c}(\mathbf{x}))\}$. Thus, we must have

$$(|\mu^A((h,t))|)_{t \in S^h} = p(\varpi(\mathsf{c}(\mathbf{x}))) \trianglerighteq_h v,$$

as when allocating capacity under p, the hospital h could have chosen the distribution vector v but instead chose the distribution vector corresponding to μ^A. Thus, we find that $(|\mu^A((h,t))|)_{t \in S^h} \trianglerighteq_h v$, as required by Definition 1.

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