Hidden Substitutes*

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Abstract

In this paper, we show that preferences exhibiting some forms of complementarity in fact have an underlying substitutable structure. In the setting of many-to one matching with contracts, we identify "hidden" substitutabilities in agents' preferences; this makes stable and strategy-proof matching possible in new settings with complementarities, even though stable outcomes are not guaranteed, in general, when complementarities are present. Our results give new insight into a range of market design settings, including the allocation of teachers to traineeships in Germany.

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1 Introduction

Imagine a world with two doctors: Sherlock (S) and Watson (W). Sherlock is a genius who can take any job in a hospital—he can do research work (r) or clinical work (c). Watson, meanwhile, is merely competent—he can only do clinical tasks, and cannot do them as well as Sherlock can. A London hospital would like to have both a researcher and a clinician, but prefers to have a clinician if only one doctor is available. Thus, the hospital's preferences over contracting outcomes take the form

$$\{\mathsf{S}^r, \mathsf{W}^c\} \succ \{\mathsf{S}^c\} \succ \{\mathsf{W}^c\} \succ \{\mathsf{S}^r\} \succ \varnothing,$$

where d^α denotes a contract under which doctor d performs role α. Note that the hospital will reject S^r when only S^r and S^c are available, but will accept S^r whenever W^c is available. Thus, there is some *complementarity* between contracts S^r and W^c: the availability of W^c makes S^r more desirable, relative to S^c. Complementarities like this are often problematic for market design, because stable equilibria typically do not exist in the presence of complementarities.[1]

But just as in a good mystery novel, not everything in our Sherlock–Watson example is as it seems. A more complete view of the hospital's preferences recognizes that *the hospital would most prefer to hire two Sherlocks*. As Sherlock is better at clinical work than Watson, the hospital's underlying preferences take the form

$$\{\mathsf{S}^r, \mathsf{S}^c\} \succ \{\mathsf{S}^r, \mathsf{W}^c\} \succ \{\mathsf{S}^c\} \succ \{\mathsf{W}^c\} \succ \{\mathsf{S}^r\} \succ \varnothing,$$

where $\{\mathsf{S}^r, \mathsf{S}^c\}$ represents an outcome in which Sherlock is assigned both jobs. Even a genius doctor like Sherlock can only hold one job at a time; thus, the hospital's most preferred outcome, $\{\mathsf{S}^r, \mathsf{S}^c\}$, is infeasible. Nevertheless, accounting for the hospital's preference for the infeasible outcome $\{\mathsf{S}^r, \mathsf{S}^c\}$ makes the apparent complementarity between S^r and W^c vanish.

[1] Preferences without complementarities (i.e., *substitutable* preferences) are necessary to guarantee the existence of stable outcomes in the settings of many-to-one matching (Hatfield and Kojima, 2008), many-to-one matching with transfers (Gul and Stacchetti, 1999; Hatfield and Kojima, 2008), many-to-many matching with and without contracts (Hatfield and Kominers, 2016), matching in vertical networks (Hatfield and Kominers, 2012), and matching in trading networks with transfers (Hatfield et al., 2013).

Indeed, now the availability of W^c has no impact on the desirability of S^r relative to S^c—whenever both S^r and S^c are available, the hospital chooses $\{\mathsf{S}^r, \mathsf{S}^c\}$. In a formal sense, the hospital's underlying preferences over contracting outcomes exhibit no complementarities at all—in the terminology of matching theory, they are *substitutable* (Kelso and Crawford, 1982; Hatfield and Milgrom, 2005). The underlying substitutability of the hospital's preferences is hidden, however, when we project the hospital's preferences into a "many-to-one matching with contracts" model that permits at most one contract with each doctor.

In this paper, we show that identifying and accounting for hidden substitutability in preferences enables new applications of the many-to-one matching with contracts framework (Kelso and Crawford, 1982; Fleiner, 2003; Hatfield and Milgrom, 2005). In particular, we highlight how we can understand the effectiveness of the German teacher traineeship allocation process by recognizing substitutable structure underlying schools' preferences, despite the presence of complementarities. Meanwhile, other authors have recently used our results to redesign the Israeli Psychology Masters Match (Hassidim et al., 2016a,b), to propose a redesign of the procedure to allocate students across the Indian Institutes of Technology (Aygün and Turhan, 2016), and to suggest a new mechanism for centralized university admissions in the United States (Yenmez, 2016).[2]

Stability and strategy-proofness are key goals of practical market design; the former guarantees a form of fairness (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003) and reduces both unraveling and *ex post* renegotiation (Roth, 1984, 1990; Kagel and Roth, 2000), while the latter eliminates the gains to strategic manipulation, both simplifying the participants' problem and ensuring that allocations are calculated using accurate preference data.[3] Moreover, guarantees about stability and strategy-proofness must be made upfront,

[2]Additionally, as we discuss in Appendix F, our results allow us to clarify why stable, strategy-proof many-to-one matching with contracts is possible in existing applications with complementarities, such as cadet–branch matching (Sönmez and Switzer, 2013; Sönmez, 2013), airline upgrade allocation (Kominers and Sönmez, 2015), and the design of affirmative action mechanisms (Kominers and Sönmez, 2015).

[3]Truthful reporting is a dominant strategy under strategy-proofness, and so strategy-proof mechanisms elicit—and thus base assignment upon—true preferences in equilibrium. Additionally, by making truthful reporting dominant, strategy-proofness eliminates the gains from strategic sophistication, thus ensuring "equal access" to the mechanism (Pathak and Sönmez, 2008).

as market mechanisms must be established prior to preference elicitation. Thus, much of the theoretical work in market design has focused on characterizing when stability and strategy-proof matching can be guaranteed, often finding that ruling out complementarities is essential.[4]

Our formalization of hidden substitutability enables us to identify a large set of choice function profiles with complementarities that can be "completed" into substitutable choice function profiles. When all choice functions in a many-to-one matching market are substitutably completable, we can use results from the theory of many-to-many matching with contracts under substitutable preferences (Hatfield and Kominers, 2016) to show the existence of stable outcomes. These observations give an intuitive illustration of why, contrary to a claim of Hatfield and Milgrom (2005), substitutability is *not* required for the guaranteed existence of stable outcomes in many-to-one matching with contracts.[5] Finally, we show that when all choice functions have substitutable completions that satisfy the Law of Aggregate Demand, there exists a deferred acceptance matching mechanism that is stable and strategy-proof for the side of the market on which all agents have unit demand.[6]

Prior to our work, a number of authors have examined structured forms of complementarity in matching contexts. Ostrovsky (2008), for example, examined supply chain matching settings in which preferences exhibit "cross-side complementarities," in the sense that opportunities to purchase inputs are complementary with opportunities to sell outputs. Cross-side complementarity, however, is really a type of substitutability condition (see Hatfield et al. (2013, 2015)).[7] Sun and Yang (2006, 2009) examine a second type of complementarity, in

[4]Substituable preferences are necessary to guarantee the existence of stable outcomes in a number of settings (see Footnote 1).

[5]Hatfield and Milgrom (2005) claimed (p. 921) that stable many-to-one matching with contracts outcomes are not guaranteed outside the domain of substitutable preferences. This claim was first found to be in error by Hatfield and Kojima (2008).

[6]Recently, Jagadeesan (2016b) has extended the work presented here by introducing a refinement of substitutable completability. Under Jagadeesan's (2016b) refinement, we recover not only stability and strategy-proofness, but also versions of the classical lattice structure and rural hospitals results.

[7]Cross-side complementarity requires that when an agent loses the opportunity to purchase an input good, it becomes less desirable for that agent to sell an output good, or, equivalently, "owning" the output good becomes more desirable for that agent; consequently, cross-side complementarity over contracts corresponds to substitutability over underlying goods in the economy.

which there are two groups of objects: objects are substitutable within groups but complementary across groups. As Hatfield et al. (2013) showed, however, it is possible to relabel the Sun and Yang (2006, 2009) market so that its complementarity structure exactly corresponds to cross-side complementarity in the sense of Ostrovsky (2008). Our work, unlike that of Ostrovsky (2008) and Sun and Yang (2006, 2009), deals with settings in which agents have choice functions that do not correspond to substitutable preferences; rather, the substitutably completable preferences we identify admit stable outcomes despite having fundamental complementarities. Our work is thus most closely related to the work of Hatfield and Kojima (2010), who introduced two other weakened substitutability conditions that ensure the existence of stable outcomes: substitutable completability subsumes the first of Hatfield and Kojima's (2010) conditions (unilateral substitutability; see Section 5.4 and Kadam (2015)) and is independent of the second (bilateral substitutability, which guarantees the existence of stable outcomes but does not ensure that stable, strategy-proof matching is possible).[8] In particular, the principal applications we discuss here, such as teacher traineeship allocation and the Israeli Psychology Masters Match, fail all of the substitutability conditions introduced in prior work.

The remainder of this paper is organized as follows: Section 2 presents the model of many-to-one matching with contracts. Section 3 defines substitutable completability and presents our main results. Section 4 shows how our results can be used to understand the mechanism for allocating teacher traineeships in Germany. Section 5 discusses how others have applied our theory of substitutable completability in a range of market design settings. Finally, Section 6 concludes. Proofs and examples omitted from the main text are presented in the Appendix.

[8]Figure 1 in Appendix D shows the relationship between substitutable completability and the substitutability structures introduced in this prior literature.

2 Model

We work with the Hatfield and Milgrom (2005) *many-to-one matching with contracts* model, in which doctors and hospitals match to each other while negotiating contractual terms. There is a finite set D of *doctors*, a finite set H of *hospitals*, and a finite set T of *contractual relationships*.[9] A *contract* $x = (d, h, t)$ is a triple specifying a doctor d, a hospital h, and a contractual relationship t. The set of all possible contracts, which we denote X, is then a subset of $D \times H \times T$.

For any set of contracts $Y \subseteq X$ and any doctor $d \in D$, we let Y_d denote the set of contracts associated with d, i.e., $Y_d \equiv \{(\bar{d}, h, t) \in Y : \bar{d} = d\}$. Similarly, for any set of contracts $Y \subseteq X$ and any $h \in H$, we let Y_h denote the set of contracts associated with h, i.e., $Y_h \equiv \{(d, \bar{h}, t) \in Y : \bar{h} = h\}$.

Each agent i has a *choice function* C^i that specifies, for any given set of contracts Y, the set of contracts i desires from Y. We require that each agent i only choose contracts associated with i, i.e., $C^i(Y) \subseteq Y_i$. Moreover, doctors have *unit demand*, i.e., for all doctors d and all sets of contracts Y, d's choice from Y, $C^d(Y)$, contains at most one contract. Hospitals, meanwhile, may demand multiple contracts. We say that the choice function of a hospital is *many-to-one* if it only selects sets of contracts that contain at most one contract with each doctor. A *profile* of choice functions is a vector $C = (C^i)_{i \in D \cup H}$.

Except where explicitly noted otherwise, we only consider choice functions that satisfy the *irrelevance of rejected contracts* condition of Aygün and Sönmez (2013, 2014); this condition is an "independence of irrelevant alternatives" condition that requires that the set of contracts an agent chooses does not change when that agent loses access to a contract not in that chosen set.[10] Formally, a choice function C^i satisfies the irrelevance of rejected contracts condition if, for all $Y \subseteq X$ and $z \in X \setminus Y$, whenever $z \notin C^i(Y \cup \{z\})$, we have $C^i(Y \cup \{z\}) = C^i(Y)$. We say that a profile of choice functions C satisfies the irrelevance of rejected contracts

[9] In practice, a contractual relationship can encode terms such as wages, work hours, and responsibilities.

[10] In particular, we assume throughout that all doctors' choice functions satisfy the irrelevance of rejected contracts condition.

condition if C^h satisfies the irrelevance of rejected contracts condition for each $h \in H$.

It is often easier to think of choice functions as derived from underlying preference relations over sets of contracts (as opposed to thinking of the choice function itself as a primitive). A *preference relation* for i, denoted $\succ_i$, is an ordering over subsets of X_i; we say that i prefers Y to $\hat{Y}$ if $Y \succ_i \hat{Y}$. A preference relation $\succ_i$ for i *induces* a choice function C^i for i, under which i chooses the subset of Y that is highest-ranked according to the preference relation $\succ_i$; that is

$$C^i(Y) = \max_{\succ_i}\{Z \subseteq X_i : Z \subseteq Y\},$$

where $\max_{\succ_i}$ indicates maximization with respect to the ordering $\succ_i$.[11] More generally, for doctors, we say that d *prefers* x to y if d chooses x over y, i.e., $\{x\} = C^d(\{x, y\})$.

2.1 Outcomes

In our framework, an *outcome* is just a set of contractual obligations for each agent; hence, an outcome can be specified by a set of contracts $Y \subseteq X$. The central equilibrium concept of matching theory is *stability*, which imposes two conditions on outcomes: First, a stable outcome Y must be *individually rational* for each agent, in the sense that no agent wishes to unilaterally abrogate any of his contracts in Y; formally, Y is individually rational under C if $C^i(Y) = Y_i$ for all $i \in D \cup H$. Second, a stable outcome Y must be *unblocked*, in the sense that no hospital and set of doctors can "block" Y by negotiating new contracts outside of Y (while possibly dropping some of the contracts in Y); formally, Y is unblocked under C if there does not exist a hospital h and a nonempty set $Z \subseteq X_h \setminus Y$ such that $Z_i \subseteq C^i(Y \cup Z)$ for all i associated to contracts in Z.

2.2 Conditions on Choice Functions

Much of matching theory depends heavily on the assumption that agents' choice functions are *substitutable*, in the sense that gaining a new offer x can not make i choose a contract z

[11] Note that any choice function induced by a preference relation automatically satisfies the irrelevance of rejected contracts condition.

that i would otherwise reject. In other words, substitutability requires that no two contracts are "complements," in the sense that access to a contract x makes a rejected contract z desirable. Formally, the choice function C^i of i is substitutable if for all $x, z \in X$ and $Y \subseteq X$, if $z \notin C^i(Y \cup \{z\})$, then $z \notin C^i(\{x\} \cup Y \cup \{z\})$. In our framework, doctors' choice functions are always substitutable because doctors have unit demand.[12,13]

The *Law of Aggregate Demand*, first introduced by Hatfield and Milgrom (2005), is a monotonicity condition that requires that if the set of contracts available to an agent expands, then that agent chooses (weakly) more contracts. Formally, a choice function C^i satisfies the Law of Aggregate Demand if for all $\hat{Y} \subseteq Y \subseteq X$, we have $|C^i(\hat{Y})| \leq |C^i(Y)|$.[14]

3 Substitutable Completability

Standard many-to-one matching with contracts models impose a requirement that each hospital's choice function be *many-to-one*—recall that a choice function is many-to-one if it selects sets of contracts that contain at most one contract with each doctor. However, as our Sherlock–Watson example in the Introduction illustrates, a hospital may have a many-to-one choice function that reflects a "true" underlying desire to assign a single doctor to multiple positions, even if the hospital is aware that the doctor demands at most one contract.

In fact, we can think of a hospital's many-to-one choice function as a projection of a more "complete" choice function that allows a hospital to choose sets of contracts that contain multiple contracts with the same doctor.

Definition. A *completion* of a many-to-one choice function C^h of hospital $h \in H$ is a choice function $\bar{C}^h$ such that for all $Y \subseteq X$, either

[12] Note that this observation also depends crucially on our assumption that doctors' choice functions satisfy the irrelevance of rejected contracts condition.

[13] A choice function C^h satisfies both the substitutability condition and the irrelevance of rejected contracts condition if and only if it is *path-independent*, i.e., if for every $Y, Z \subseteq X$, we have that $C^h(Y \cup Z) = C^h(Y \cup C^h(Z))$. The linkage between path independence and our key conditions was first noted by Aizerman and Malishevski (1981); Chambers and Yenmez (2013) recently extended this observation to matching with contracts.

[14] Alkan and Gale (2003) introduced a related condition called *size monotonocity*.

- $\bar{C}^h(Y) = C^h(Y)$, or
- there exist distinct $z, \hat{z} \in \bar{C}^h(Y)$ that are associated with the same doctor, i.e., $z, \hat{z} \in X_d$ for some $d \in D$.

We say that a profile of choice functions $\bar{C}$ is a *completion* of a profile of choice functions C if, for each hospital $h \in H$, the choice function $\bar{C}^h$ is a completion of the associated choice function C^h, and $\bar{C}^d = C^d$ for each doctor $d \in D$.

Note that every choice function is a completion of itself. Moreover, it is generally the case that nontrivial completions of a hospital's choice function exist. For an example, consider the choice function C^{h} of hospital h induced by the preference relation given in the Sherlock–Watson example in the Introduction,

$$\succ_{\mathsf{h}} : \{\mathsf{S}^r, \mathsf{W}^c\} \succ \{\mathsf{S}^c\} \succ \{\mathsf{W}^c\} \succ \{\mathsf{S}^r\} \succ \varnothing.$$

Note that the choice function C^{h} induced by the preference relation $\succ_{\mathsf{h}}$ is not substitutable, as $\mathsf{S}^r \notin \{\mathsf{S}^c\} = C^{\mathsf{h}}(\{\mathsf{S}^r, \mathsf{S}^c\})$, while $\mathsf{S}^r \in \{\mathsf{S}^r, \mathsf{W}^c\} = C^{\mathsf{h}}(\{\mathsf{S}^r, \mathsf{S}^c, \mathsf{W}^c\})$.

A natural completion of the choice function C^{h} is

$$\bar{C}^{\mathsf{h}}(Y) \equiv \begin{cases} \{\mathsf{S}^r, \mathsf{S}^c\} & \{\mathsf{S}^r, \mathsf{S}^c\} \subseteq Y \\ C^{\mathsf{h}}(Y) & \text{otherwise.} \end{cases} \tag{1}$$

The completion $\bar{C}^{\mathsf{h}}$ uncovers the "hidden" substitutability of the choice function C^{h}: Under $\bar{C}^{\mathsf{h}}$, the hospital would be willing to choose Sherlock as both its clinician and its researcher if Sherlock were willing to do both tasks. Indeed, the completed choice function $\bar{C}^{\mathsf{h}}$ is induced by the preference relation

$$\{\mathsf{S}^r, \mathsf{S}^c\} \succ \{\mathsf{S}^r, \mathsf{W}^c\} \succ \{\mathsf{S}^c\} \succ \{\mathsf{W}^c\} \succ \{\mathsf{S}^r\} \succ \varnothing$$

described in the Introduction. Consequently, unlike the original choice function C^{h}, the completed choice function $\bar{C}^{\mathsf{h}}$ is substitutable—in particular, $\mathsf{S}^r \in \{\mathsf{S}^r, \mathsf{S}^c\} = \bar{C}^{\mathsf{h}}(\{\mathsf{S}^r, \mathsf{S}^c\})$.

Looking at substitutable completions allows us to find stable outcomes in the setting of many-to-one matching with contracts for certain types of non-substitutable choice functions.

To see this, we begin by relating the set of outcomes stable under the profile of choice functions C to the set of outcomes stable under a given completion $\bar{C}$ of C.

Lemma 1. *If $\bar{C}$ is a completion of a profile of choice functions C, and $\bar{C}$ satisfies the irrelevance of rejected contracts condition, then any outcome stable with respect to $\bar{C}$ is stable with respect to C.*

The proof of Lemma 1 begins by observing that if Y is a stable outcome under $\bar{C}$, then, in particular, Y is individually rational for each doctor under $\bar{C}^D = C^D$. Because doctors have unit demand, individual rationality for doctors implies that Y contains at most one contract associated with each doctor. Consequently, for each hospital h, we must have that $\bar{C}^h(Y) = C^h(Y)$ (as there is no possibility that $\bar{C}^h(Y) \subseteq Y$ contains two contracts with any single doctor); individual rationality of Y for hospitals under C^h then follows from individual rationality of Y under $\bar{C}$. Finally, if Y is blocked by Z under C and h is the hospital associated with Z, then (by the definition of completion) either $\bar{C}^h(Y \cup Z) = C^h(Y \cup Z)$ or $\bar{C}^h(Y \cup Z)$ contains at least two contracts with some doctor d. In the former case, Z blocks Y under $\bar{C}$ directly. In the latter case, we know that $W \equiv [\bar{C}^h(Y \cup Z)] \setminus Y$ is nonempty, as Y contains at most one contract with d (and $\bar{C}^h(Y \cup Z)$ contains at least two); the set W then blocks Y under $\bar{C}$.[15]

If a choice function C^h has a completion that is substitutable, then we say that C^h is *substitutably completable*.[16] If every choice function in a profile of choice functions C is substitutably completable, then we say that C is *substitutably completable*. The remainder of this section demonstrates that substitutably completable choice functions inherit many useful properties from their completions.

In essence, we can think of a substitutably completable choice function C^h as the many-to-one projection of a substitutable choice function $\bar{C}^h$ from a richer choice function space. We

[15] The observation that W blocks Y follows from the substitutability of $\bar{C}^h$ and the fact that $W \subseteq Z \subseteq \bar{C}^h(Y \cup Z)$ implies that $W \subseteq \bar{C}^h(Y \cup W)$.

[16] Note that as every choice function is a completion of itself, all substitutable choice functions are trivially substitutably completable.

may view the completion $\bar{C}^h$ of a many-to-one choice function C^h as a choice function in the setting of *many-to-many matching with contracts*, in which doctors, as well as hospitals, may demand multiple contracts. As substitutable choice functions (that satisfy the irrelevance of rejected contracts condition) are sufficient to guarantee the existence of stable outcomes in many-to-many matching with contracts (Hatfield and Kominers, 2012, 2016), we see that substitutable completability is sufficient to guarantee the existence of a stable outcome:[17] If $\bar{C}$ is a substitutable completion of a profile of choice functions C (and moreover $\bar{C}$ satisfies the irrelevance of rejected contracts condition), then there exists at least one outcome that is stable with respect to $\bar{C}$—and hence, stable with respect to C by Lemma 1.[18]

Theorem 1. *If the profile of choice functions C has a substitutable completion that satisfies the irrelevance of rejected contracts condition, then there exists an outcome that is stable with respect to C.*

Hatfield and Milgrom (2005, p. 921) claimed—incorrectly—that substitutability of hospitals' choice functions is necessary to ensure the existence of stable outcomes in many-to-one matching with contracts.[19] However, Theorem 1 identifies a natural class of many-to-one

[17] In our work on contract language design in many-to-many matching with contacts (Hatfield and Kominers, 2016), we consider a many-to-many matching model in which two given agents are allowed to sign multiple contracts with each other. We argue there that allowing two given agents to sign multiple contracts with each other is valuable for modeling *many-to-many* matching with contracts, in part because it enables substitutable representations of some types of preferences. Our exercise here is different, however: we combine our substitutable completabilility insight with the Hatfield and Kominers (2016) result on stable outcomes in many-to-many matching with contracts settings in order to find stable outcomes in some *many-to-one* matching with contracts markets in which agents have preference complementarities.

[18] In a sense, our work here hearkens back to the work of Fleiner (2003), who introduced a model of matching with contracts that does not sharply distinguish between many-to-one and many-to-many matching. Fleiner (2003) showed the existence of stable outcomes in his setting via Tarski's fixed-point theorem, building on an insight of Adachi (2000); in particular, Fleiner's (2003) work shows that the fixed-point approach does not depend, in principle, on whether hospitals are allowed to demand multiple contracts with a given doctor.

Our approach shows that passing between many-to-one and many-to-many matching is a helpful way to think about matching with contracts. However, our work also shows that simply treating the many-to-one model as a many-to-many model in which one side simply happens to have unit-demand preferences is an incomplete approach, as the many-to-one model has structure not present in the many-to-many model. Indeed, we sometimes need to transform hospitals' choice functions as we move from many-to-one matching with contracts to many-to-many matching with contracts in order to show the existence of stable outcomes. For instance, in a setting in which hospitals have preferences like those in the Sherlock–Watson example, results from many-to-many matching with contracts do not imply the existence of stable outcomes until hospitals' preferences are substitutably completed.

[19] Hatfield and Kojima (2008) first identified the error of Hatfield and Milgrom (2005).

choice functions with complementarities for which stable outcomes are guaranteed—the class of substitutably completable choice functions. Thus, with the benefit of hindsight, our Theorem 1 provides an intuitive reason why many-to-one substitutability is not strictly necessary for stability: while substitutability is key for the existence of stable outcomes, sometimes non-substitutable many-to-one choice functions can inherit substitutable behavior from many-to-many choice functions.

We illustrate Theorem 1 via an extension of our Sherlock–Watson example. Recall that the choice function C^{h} of the hospital h is induced by the preference relation

$$\succ_{\mathsf{h}} : \{\mathsf{S}^r, \mathsf{W}^c\} \succ \{\mathsf{S}^c\} \succ \{\mathsf{W}^c\} \succ \{\mathsf{S}^r\} \succ \varnothing.$$

Consider the choice function C^{S} for Sherlock induced by the preference relation

$$\succ_{\mathsf{S}} : \{\mathsf{S}^c\} \succ \{\mathsf{S}^r\} \succ \varnothing$$

and the choice function C^{W} for Watson induced by the choice function

$$\succ_{\mathsf{W}} : \{\mathsf{W}^c\} \succ \varnothing.$$

Now consider the choice function $\bar{C}^{\mathsf{h}}$ defined in (1). It is straightforward to compute that there is only one outcome stable under the profile $\bar{C} = (\bar{C}^{\mathsf{h}}, C^{\mathsf{S}}, C^{\mathsf{W}})$: the outcome $\{\mathsf{S}^c\}$. By Lemma 1, we see that $\{\mathsf{S}^c\}$ is stable under C; this can also be checked directly.

The Sherlock–Watson example also illustrates that not every outcome that is stable under a profile of choice functions is stable for every completion of that profile: Under C, both $\{\mathsf{S}^c\}$ and $\{\mathsf{S}^r, \mathsf{W}^c\}$ are stable, while under $\bar{C}$, only $\{\mathsf{S}^c\}$ is stable.

Although Lemma 1 shows that any outcome stable under a completion of C must also be stable under C, different completions of C may yield different sets of stable outcomes (see Appendix B). That said, there is a distinguished outcome that is stable under *every* completion of C: the result of the *(doctor-proposing) cumulative offer process*.

For a set of contracts Y, we say that x is the *most-preferred* contract from Y for d under C if d chooses $\{x\}$ from Y under C, i.e., $\{x\} = C^d(Y)$; with this definition, the doctor-proposing cumulative offer process under C proceeds as follows.

Step 1: Each doctor *proposes* his most-preferred contract from X under C (assuming there is one); the set of proposed contracts is denoted A^1. Each hospital h *holds* its favorite set of contracts from those that have been proposed, i.e., $C^h(A^1)$.

Step τ: Each doctor not associated with a currently held contract proposes his most-preferred contract that has not yet been proposed (if any), i.e., his most preferred contract from $X \setminus A^{\tau-1}$ under C. If no contract is proposed, then the algorithm terminates and the outcome is the set of contracts held by the hospitals from the set of proposed contracts, i.e., $\bigcup_{h \in H} C^h(A^{\tau-1})$. Otherwise, the set of contracts proposed in Steps 1 through τ is denoted A^τ; each hospital h holds its favorite set of contracts from those that have been proposed, i.e., $C^h(A^\tau)$; and the algorithm proceeds to Step $\tau+1$.

The cumulative offer process was first introduced by Kelso and Crawford (1982) in a many-to-one matching with salaries model; the form we use here was introduced by Hatfield and Milgrom (2005). Note that the cumulative offer process in principal allows hospitals to hold contracts from A^τ that are not held in a prior step $\hat{\tau}$ of the algorithm (where $\hat{\tau} < \tau$). However, when all hospitals' choice functions are substitutable (and satisfy the irrelevance of rejected contracts condition), hospitals never "take back" contracts that were available but not held at some prior step. Consequently, when all hospitals' choice functions are substitutable (and satisfy the irrelevance of rejected contracts condition), the doctor-proposing cumulative offer process is equivalent to a *deferred acceptance process* under which, in Step τ, each hospital is allowed to hold only contracts that were either held by that hospital in Step $\tau - 1$ or newly proposed in Step τ.[20]

As the logic of the previous paragraph only depends on the substitutability of choice functions, it is also true for any substitutable completion $\bar{C}$ of C that $\bar{C}$ never "takes back" any contract that is available but not held at some prior step. Moreover, since the doctors who propose in any given step are a subset of those doctors without a held contract, at every step of a cumulative offer process under a substitutable completion $\bar{C}$, each hospital holds at

[20] For completeness, we add a formal description of the deferred acceptance process in Appendix C.

most one contract with each doctor (as the only newly-available contracts are with doctors not currently held); hence, by the definition of a completion, at every step the behavior of $\bar{C}$ is the same as the behavior as C. This logic implies that the path of the cumulative offer process under some completion $\bar{C}$ is the same as the path of the cumulative offer process under C.

Meanwhile, as demonstrated by Hatfield and Kominers (2012, 2016), when all agents' choice functions are substitutable (and satisfy the irrelevance of rejected contracts condition), the cumulative offer process yields a stable outcome. The preceding observations imply our next result.

Theorem 2. *If C has a substitutable completion that satisfies the irrelevance of rejected contracts condition, then the outcome of the doctor-proposing cumulative offer process under C is the same as the outcome of the doctor-proposing cumulative offer process under any substitutable completion of C; moreover, that outcome is stable under C.*

Theorem 2 implies that, when hospitals' choice functions are substitutably completable (in a way that satisfies the irrelevance of rejected contracts condition), one does not need to compute substitutable completions in order to find stable outcomes—it is sufficient to run the cumulative offer process using the hospitals' original choice functions.[21]

Another consequence of substitutable completability is that, under the Law of Aggregate Demand, the cumulative offer process makes truth-telling a dominant strategy, just as it does under substitutable choice functions (Hatfield and Milgrom, 2005).[22] We say that the cumulative offer process is *strategy-proof (for doctors)* if no doctor can obtain a strictly-preferred outcome by misreporting his choice function; that is, each doctor d weakly prefers the contract he obtains (if any) in the outcome generated by the cumulative offer process

[21] Additionally, Theorem 2 implies that the outcome of the cumulative offer process is, in a sense, canonical: if C has a substitutable completion that satisfies the irrelevance of rejected contracts condition, then the cumulative offer process produces the same outcome regardless of which substitutable completion we use, and, moreover, that same outcome is produced by running the cumulative offer process using the hospitals' original choice functions.

[22] Once again, here we require the irrelevance of rejected contracts condition.

under the profile C to the contract he obtains (if any) in the outcome generated by the cumulative offer process under any other profile of the form $(\tilde{C}^d, C^{D \setminus \{d\}}, C^H)$.

Theorem 3. *If, for each $h \in H$, the choice function C^h has a substitutable completion that satisfies the Law of Aggregate Demand and the irrelevance of rejected contracts condition, then the cumulative offer process is strategy-proof (for doctors).*[23,24]

4 Application: Assigning Teacher Traineeships

In this section, we illustrate how our results on substitutable completability can be used in the design of practical matching mechanisms. We focus on an example application: the allocation of teachers to schools. Teacher allocation has been outside the scope of previous matching frameworks because—as we demonstrate in the sequel—schools' preferences over teacher assignments are not necessarily substitutable.[25]

Specifically, we show how our results can be used to understand the mechanism for allocating teacher traineeships in Germany. In Germany, prospective teachers must complete a traineeship in order to take the Second State Exam and receive certification. Within each Bundesland, traineeships are allocated by a centralized mechanism; approximately 30,000 teachers are assigned traineeships each year. Here, we concentrate on the market for teacher traineeship positions at *gymnasia*, German secondary education schools that prepare students for college.

In the German teacher traineeship market, teacher traineeship candidates rank school-subject pairs and schools rank candidates on the basis of test scores, GPAs, waiting times, and "social hardship" (such as responsibility for a child); however, a school may designate some positions as reserved for teachers willing to teach certain subjects, such as chemistry

[23] Indeed, our proof of Theorem 3 shows a stronger result: under the assumptions of Theorem 3, the cumulative offer process is *group strategy-proof (for doctors)*, in the sense that no coalition of doctors can make each doctor in the coalition strictly better off by jointly misreporting their choice functions.

[24] We discuss the relationship between subsitutable completablity and classical structural results (such as lattice structure and the rural hospital theorem) in Appendix E (see also Jagadeesan (2016b)).

[25] Moreover, schools' choice functions fail the unilateral substitutability condition of Hatfield and Kojima (2010), and are not a special case of the slot-specific priorities structure of Kominers and Sönmez (2014).

and mathematics.

For a simple example of the flexibility of school preferences allowed by the German teacher traineeship system, we consider a setting with one school, *Freie Berufsbildende Integrationsschule* (FBI), two candidates, Mulder (M) and Scully (S), and two subjects, astronomy (a) and biology (b). The school FBI has two open positions: one position is only for an astronomy teacher (which is the highest priority to fill), while the second position would accept a teacher in either subject, but would prefer a biology teacher. Moreover, the school prefers to have Mulder teach astronomy while Scully teaches biology. Hence, we can write the preferences of FBI as

$$\succ_{\mathsf{FBI}} : \{\mathsf{M}^a, \mathsf{S}^b\} \succ \{\mathsf{M}^a, \mathsf{S}^a\} \succ \{\mathsf{M}^b, \mathsf{S}^a\} \succ \{\mathsf{M}^a\} \succ \{\mathsf{S}^a\} \succ \{\mathsf{S}^b\} \succ \{\mathsf{M}^b\} \succ \varnothing, \tag{2}$$

where, as in our Sherlock–Watson example, we use the notation d^α to denote a contract under which d performs role α. The choice function induced by the preferences of FBI is not substitutable, as S^b is not chosen by FBI when $\{\mathsf{S}^a, \mathsf{S}^b\}$ is available, but S^b is chosen by FBI when $\{\mathsf{M}^a, \mathsf{S}^a, \mathsf{S}^b\}$ is available.[26] Consequently, prior to our present work, we would not be able to infer that stable and strategy-proof matching is possible in the presence of preferences like those of FBI.

However, the truth is that a substitutable completion is out there: if we think of the school as able to choose multiple contracts with each candidate, we can interpret its underlying preferences as

$$\{\mathsf{M}^a, \mathsf{S}^b\} \succ \{\mathsf{M}^a, \mathsf{M}^b\} \succ \{\mathsf{S}^a, \mathsf{S}^b\} \succ \{\mathsf{M}^a, \mathsf{S}^a\} \succ \{\mathsf{M}^b, \mathsf{S}^a\} \succ \{\mathsf{M}^a\} \succ \{\mathsf{S}^a\} \succ \{\mathsf{S}^b\} \succ \{\mathsf{M}^b\} \succ \varnothing, \tag{3}$$

which *do* induce a substitutable choice function. The key insight is that because a given teacher traineeship candidate can only be employed to teach one subject, a school may wish to assign a teacher it currently employs to a new subject when some other candidate proposes a contract. In the example, the school wishes to reassign Scully from astronomy to biology

[26] Moreover, the choice function of FBI fails the unilateral substitutability condition of Hatfield and Kojima (2010).

once Mulder is available to teach astronomy.[27] The existence of a substitutable completion of the choice function C^{FBI} (that satisfies the irrelevance of rejected contracts condition) enables us to guarantee the existence of stable outcomes even given the complementarities in FBI's preferences; moreover, as the substitutable completion induced by (3) satisfies the Law of Aggregate Demand, the cumulative offer process will be stable and strategy-proof.

Our Proposition 1 below builds on the preceding ideas to show that, in general, the choice functions of schools in the German teacher traineeship market have substitutable completions that satisfy the Law of Aggregate Demand and the irrelevance of rejected contracts condition. Thus, our main results imply that the German teacher traineeship allocation mechanism is stable and strategy proof.

We now formally model the German teacher traineeship market. We take the set D to represent the set of teacher traineeship candidates, the set H to represent the set of schools, and the set T to represent the set of subjects candidates can teach (these are the relevant contractual terms in this setting). Each school $h \in H$ has a set of *subject-specific positions* P^h, with each $p \in P^h$ associated to a specific subject $\mathsf{t}(p) \in T$; these positions represent traineeship openings which the school wishes to fill with teachers in specific subjects. Moreover, each school $h \in H$ has, for each subject $t \in T$, a *subject-specific ranking* $\succ_{(h,t)}$ over contracts for that subject and the *null contract* $\emptyset$, i.e., over contracts in $\{(\bar{d}, h, t) \in X : \bar{d} \in D\} \cup \{\emptyset\}$. Each school $h \in H$ is endowed with a *precedence ordering* $\rhd_h^Y$ over positions that determines, as a function of the set of proposed contracts Y, the order in which positions will be filled.

The school h also has an overall quota q^h, which is weakly larger than the number of subject-specific positions (i.e., $q^h \geq |P^h|$). After h fills as many of it subject-specific positions as possible, it hires teacher traineeship candidates up to its quota q^h; for this, h ranks contracts according to a *general contract ranking* $\succ_{(h,\star)}$ over all contracts with that school (and the null contract), i.e., $\{(\bar{d}, h, \bar{t}) \in X : \bar{d} \in D \text{ and } \bar{t} \in T\} \cup \{\emptyset\}$.

[27] In fact, in one of the authors' casual conversations with his department chair, it came up that similar issues arise when allocating lecturers to core and elective MBA classes.

The choice function C^h of a school h is constructed according to the following rule: For an available set of contracts Y,

1. Initialize the set of *available contracts* as $A^0 = Y$ and the set of *selected contracts* as $G^0 = \varnothing$.

2. Label the positions in P^h as $p^1, p^2, \ldots, p^{|P^h|}$, where p^ℓ is the ℓ^{th} highest position according to the precedence order $\rhd_h^Y$.

3. For each $\ell = 1, \ldots, |P^h|$, let x^ℓ be the highest-ranked contract in the set of available contracts $A^{\ell-1} \cup \{\emptyset\}$ according to h's ranking over contracts for the subject $\mathsf{t}(p^\ell)$, i.e., the subject specific ranking $\succ_{(h,\mathsf{t}(p^\ell))}$. If $x^\ell \neq \emptyset$, let d^ℓ be the teacher traineeship candidate associated with x^ℓ. Add x^ℓ to the set of selected contracts, i.e., let $G^\ell = \{x^\ell\} \cup G^{\ell-1}$, and remove any contracts associated with d^ℓ from the set of available contracts, i.e., $A^\ell = A^{\ell-1} \setminus Y_{d^\ell}$; otherwise (i.e., if $x^\ell = \emptyset$), let $G^\ell = G^{\ell-1}$ and $A^\ell = A^{\ell-1}$.

4. For each $\ell = |P^h| + 1, \ldots, q^h + (|P^h| - |G^{|P^h|}|)$, let x^ℓ be the highest-ranked contract in the set of available contracts $A^{\ell-1} \cup \{\emptyset\}$ according to the general contract ranking $\succ_{(h,\star)}$.[28] If $x^\ell \neq \emptyset$, let d^ℓ be the teacher traineeship candidate associated with x^ℓ. Add x^ℓ to the set of selected contracts, i.e., let $G^\ell = \{x^\ell\} \cup G^{\ell-1}$, and remove any contracts associated with d^ℓ from the set of available contracts, i.e., $A^\ell = A^{\ell-1} \setminus Y_{d^\ell}$; otherwise (i.e., if $x^\ell = \emptyset$), let $G^\ell = G^{\ell-1}$ and $A^\ell = A^{\ell-1}$.

5. Finally, take the choice of h from Y to be the set of selected contracts, i.e., $C^h(Y) = G^{q^h + (|P^h| - |G^{|P^h|}|)}$.

We say that a choice function determined by the preceding algorithm has a *gymnasium priority structure*.

Gymnasium priorities allow us to model markets in which some schools have preferences that induce complementarities such as those in the FBI example: To model the preferences of

[28] The $|P^h| - |G^{|P^h|}|$ term comes from the fact that subject-specific positions that are unfilled "revert" to general open positions.

FBI, we let the set of teachers be given by $D = \{\mathsf{M}, \mathsf{S}\}$ and let the set of subjects be given by $T = \{a, b\}$. The school has one subject-specific position p associated with astronomy (and no subject-specific positions associated with biology); the preference ordering for astronomy, $\succ_{(\mathsf{FBI},a)}$, is given by

$$\succ_{(\mathsf{FBI},a)} : \{\mathsf{M}^a\} \succ \{\mathsf{S}^a\} \succ \emptyset,$$

while the preference ordering for biology, $\succ_{(\mathsf{FBI},b)}$, is given by[29]

$$\succ_{(\mathsf{FBI},b)} : \{\mathsf{S}^b\} \succ \{\mathsf{M}^b\} \succ \emptyset.$$

Since there is only one position, the only possible precedence ordering $\rhd^Y_{\mathsf{FBI}}$ is trivial for all Y. Finally, the quota for school FBI is given by $q_{\mathsf{FBI}} = 2$ and the general contract ranking is given by

$$\succ_{(\mathsf{FBI},\star)} : \{\mathsf{S}^b\} \succ \{\mathsf{M}^b\} \succ \{\mathsf{M}^a\} \succ \{\mathsf{S}^a\} \succ \emptyset.$$

The choice function generated by the FBI gymnasium priority structure specified here is the same as the choice function induced by the preference relation $\succ_{\mathsf{FBI}}$ given by (2). Moreover, the choice function C^{FBI} is not substitutable but is substitutably completable.[30] More generally, in fact, all choice functions that have gymnasium priority structures have well-behaved substitutable completions.

Proposition 1. *Every choice function that has a gymnasium priority structure has a substitutable completion that satisfies the Law of Aggregate Demand and the irrelevance of rejected contracts condition.*

Combining Proposition 1 with Theorems 2 and 3 yields the following corollary.

Corollary 1. *If all schools have choice functions that have gymnasium priority structures, then the cumulative offer process produces a stable outcome and is strategy-proof (for teacher traineeship candidates).*

[29] Note that the biology-specific ranking can not affect the outcome, as the school has no subject-specific positions associated with biology.

[30] Recall that a substitutable completion of C^{FBI} is induced by the preference relation(3).

Corollary 1 implies that the matching generated by the German teacher traineeship allocation mechanism is stable; as documented by Roth (1991), stable mechanisms are more likely to succeed in practice. Moreover, Corollary 1 implies that the German teacher traineeship allocation mechanism is strategy-proof for the teacher traineeship candidates; this simplifies the participants' problem and, in particular, ensures "equal access" to the mechanism in the sense of Pathak and Sönmez (2008).

As we show in Appendix F.1, gymnasium priority structures are special cases of a more general class of *tasks-and-slots priority structures*—and any choice function induced by a tasks-and-slots priority structure has a substitutable completion that satisfies the Law of Aggregate Demand and the irrelevance of rejected contracts condition.[31]

5 Substitutable Completability in Other Markets

Since we first circulated this work, a number of authors have developed novel applications of substitutable completability to real-world matching problems; here, we briefly survey these applications.

5.1 The Israeli Psychology Masters Match

Hassidim et al. (2016a,b) recently redesigned the Israeli Psychology Masters Match (IPMM), which matches applicants to graduate studies in psychology in Israel. The principal goal of the IPMM redesign was to implement a mechanism that is stable and strategy-proof (for applicants).[32] However, the IPMM features a large range of contractual terms (such as

[31] Tasks-and-slots priority structures feature two different types of positions: *tasks* and *slots*. Tasks are filled before slots, and the order in which tasks are filled may depend on the set of contracts available; however, any two tasks either have identical priority orderings or find disjoint sets of contracts acceptable. Meanwhile, in principle, any contract can be accepted by any slot, but slots must be filled in a fixed sequence.

A gymnasium priority structure can be realized as a tasks-and-slots priority structure by taking the tasks to correspond to subject-specific positions, while the slots correspond to positions that are not tied to a specific subject. Additionally, tasks-and-slots priority structures generalize the *slot-specific priorities* of Kominers and Sönmez (2014, 2015). Thus, our work also provides a new proof that the cumulative offer mechanism is stable and strategy-proof under slot-specific priorities.

[32] Stability was desired in order to eliminate "unraveling" of the type observed by Roth and Xing (1994)—there were widespread beliefs that some departments coordinated amongst themselves on who would make offers to which candidates, and that other departments made more offers than they had positions available

type of degree and fellowship status), and some programs have complex preferences (such as affirmative action constraints and rules for balancing the allocation between clinical and research positions); as a result, many of the graduate programs involved in the IPMM have preferences that are not only non-substitutable, but also fail all the weakened substitutability conditions introduced prior to our work (see Hassidim et al. (2016a)). Nevertheless, in order to get the graduate programs to agree to the IPMM redesign, it was essential for the redesigned mechanism to enable programs to express preferences at their true levels of complexity. Hassidim et al. (2016a) were able to show (*after* soliciting unrestricted preference structures from the programs) that all the programs' preference structures are, in fact, substitutably completable.[33] Our results here were then used to facilitate stable and strategy-proof matching in the IPMM. The completion-based IPMM has now been successfully run for two years, with both programs and students expressing satisfaction with the process (Hassidim et al., 2016a).

5.2 College Admissions in India

Aygün and Turhan (2016) studied the allocation of over 300,000 students to the Indian Institutes of Technology (IIT). In the IIT matching mechanism, schools must set aside a certain number of slots for students from different privileged groups; however, a reserved slot may "revert" to a regular seat if it is not taken by a member of a privileged group.[34] A student from a privileged group may prefer a seat reserved for privileged groups (as such seats come with significant financial aid) but also might prefer an unreserved seat (as students who take reserved seats face discrimination on campus). Aygün and Turhan (2016) observed that the choice functions used in the IIT student matching mechanism are not substitutable; moreover, those choice functions are not examples of any of the non-

(in order to ensure they filled their quota). Roth (1991) showed that stable mechanisms have alleviated unraveling in the United States and elsewhere. Strategy-proofness was desired to simplify the strategic problem faced by applicants (Hassidim et al., 2016b).

[33]Moreover, the natural substitutable completions that Hassidim et al. (2016a) identified satisfy the Law of Aggregate Demand and the irrelevance of rejected contracts condition.

[34]The privileged groups are comprised of "scheduled castes," "scheduled tribes," and "other backward classes," groups that have been historically disadvantaged in India.

substitutable, but still well-behaved, choice function classes identified by previous work. Thus, Aygün and Turhan (2016) use our theory of substitutable completability (specifically, our Theorems 1–3) to argue that the IIT system could improve its allocation mechanism by using an implementation of the cumulative offer process.

5.3 College Admissions

Yenmez (2016) builds on our work here to propose a new approach for centralized college admissions. Yenmez (2016) treats college admissions as a many-to-many matching with contracts problem, in which students can be matched with many "admissions offers" which may include financial aid. Implementing binding "early decision" rules into college admissions introduces non-substitutabilities in colleges' choice functions; however, Yenmez (2016) shows that every college's choice function has a substitutable completion (that satisfies the irrelevance of rejected contracts condition). Yenmez (2016) then generalizes our results to his many-to-many matching with contracts setting to show the existence of stable admissions outcomes.[35]

5.4 Unilaterally Substitutable Preferences

Hatfield and Kojima (2010) introduced *unilateral substitutability*, a condition on preferences that ensures that the deferred acceptance algorithm produces a stable outcome and is strategy-proof for doctors. Unilateral substitutability has been central in the analysis of cadet–branch matching problems: Sönmez and Switzer (2013) and Sönmez (2013) showed that U.S. military branches' preferences over contracts with cadets are unilaterally substitutable (but not substitutable), and then used this observation to show the existence of a stable strategy-proof cadet–branch matching mechanism very similar to the mechanism already used by the U.S. Army.

[35]For Yenmez's (2016) results, it is essential that students recieve no more than one admissions offer from each college—a version of an assumption that Kominers (2012) calls "unitarity." In other work (Hatfield and Kominers, 2016), we have shown that in non-unitary many-to-many matching with contracts models, substitutability is necessary (in the maximal domain sense) for the existence of stable outcomes.

In fact, any unilaterally substitutable choice function induced by a preference relation is substitutably completable, as Kadam (2015) has recently shown. Thus, for many applications, substitutable completability may provide a technically simpler and more intuitive alternative to unilateral substitutability.[36]

6 Conclusion

Non-substitutable choice functions that are substitutably completable have a hidden, underlying substitutable structure: they are projections of substitutable choice functions from the broader preference domain of many-to-many matching with contracts. Because of this structure, the existence of a substitutable completion (that satisfies the Law of Aggregate Demand and the irrelevance of rejected contracts condition) guarantees that the cumulative offer process produces a stable outcome and is strategy-proof for doctors.

In the Hatfield and Milgrom (2005) formulation of many-to-one matching with contracts, the condition that each doctor is assigned at most one contract is enforced by both restricting doctors to demand at most one contract and restricting hospitals to demand at most one contract with each doctor. However, the restriction on doctor preferences is sufficient to guarantee that each doctor has only one contract in any stable outcome; this implies that the restriction on hospital preferences is (formally) unnecessary. Thus, in some sense, our approach hearkens back to the earlier matching with contracts model of Fleiner (2003), which did not formally impose the constraint that each hospital can choose at most one contract with each doctor. Substitutable completability shows that this issue is not just a theoretical curiosity; rather, if we treat each hospital as willing to accept multiple contracts with the

[36] Hatfield and Kojima (2010) also showed that stable outcomes are guaranteed to exist when hospital choice functions satisfy the weaker condition of *bilateral substitutability*. In Appendix D, we show that there exist substitutably completable choice functions that are not bilaterally substitutable (and, hence, do not satisfy the stronger condition of unilateral substitutability); there, we also show that there exist hospital choice functions that are bilaterally substitutable but are not substitutably completable. It is an open question whether there is a condition on hospital choice functions sufficient and necessary (in the maximal domain sense) to guarantee the existence of stable outcomes. In work subsequent to ours, Hatfield, Kominers, and Westkamp (2015) have identified a set of conditions that are both sufficient and necessary (in the maximal domain sense) for the guaranteed existence of stable and strategy-proof matching mechanisms.

same doctor, then we can extend the applicability of the matching with contracts model. In particular, our theory of substitutable completability enables us to achieve stable and strategy-proof matching in settings ranging from teacher allocation (Section 4) to college admissions (Aygün and Turhan, 2016; Yenmez, 2016).

Our results highlight how a deep understanding of substitutability is essential for market design. Matching with contracts depends crucially on substitutability, but recent work including ours and others' (e.g., Ostrovsky (2008), Milgrom (2009), Echenique (2012), Ostrovsky and Paes Leme (2014), and Jagadeesan (2016a,b)) shows that substitutability is subtle—indeed, it sometimes hides in plain sight.

> *"Circumstantial evidence is a very tricky thing [. . . .] It may seem to point very straight to one thing, but if you shift your own point of view a little, you may find it pointing in an equally uncompromising manner to something entirely different."*
>
> —Sherlock Holmes, in *The Boscombe Valley Mystery*

A Proofs Omitted from the Main Text

Throughout this appendix, we denote by $\mathsf{d}(x)$ the doctor associated with contract x; similarly, we denote by $\mathsf{d}(Y)$ the set of doctors associated with some contract in Y, i.e., $\mathsf{d}(Y) = \cup_{y \in Y} \mathsf{d}(y)$.

Proof of Lemma 1

We assume that A is stable with respect to $\bar{C}$, and show that A is stable with respect to C. We prove the result in three steps:

A is individually rational for doctors under C: As doctors have the same choice functions under $\bar{C}$ as under C, the individual rationality of A under $\bar{C}^d$ for each doctor $d \in D$ immediately implies the individually rationality of A under C^d for each doctor $d \in D$.

A is individually rational for hospitals under C: The individual rationality of A for doctors implies that each doctor has at most one contract in A, i.e., $|A_d| \leq 1$ for each $d \in D$. Then, as $\bar{C}^h$ completes C^h, it follows that $C^h(A) = \bar{C}^h(A)$ for all $h \in H$ as A does not contain two (or more) contracts with any individual doctor; hence, the individual rationality of A under $\bar{C}^h$ for each hospital $h \in H$ immediately implies the individually rationality of A under C^h for each hospital $h \in H$.

A is unblocked under C: Suppose that A is blocked under C by some hospital h and a blocking set $Z \subseteq X_h \setminus A$ under C. First, as Z blocks A under C, and $\bar{C}^d = C^d$ for each $d \in D$, we know that

$$Z_d \subseteq C^d(Z \cup A) = \bar{C}^d(Z \cup A) \quad \text{for all } d \in D. \tag{4}$$

Now, as $\bar{C}^h$ completes C^h, we know from the definition of completability that either

- $\bar{C}^h(Z \cup A) = C^h(Z \cup A)$, or
- there exist distinct $z, \hat{z} \in W \equiv \bar{C}^h(Z \cup A)$ such that $\mathsf{d}(z) = \mathsf{d}(\hat{z})$.

In the former case, we have $Z_h \subseteq C^h(Z \cup A) = \bar{C}^h(Z \cup A)$, as Z blocks A under C; combining this fact with (4) shows that Z blocks A under $\bar{C}$, contradicting the stability of A under $\bar{C}$.

In the latter case, we note that as A is individually rational for doctors under C, we must have $|A_{\mathsf{d}(z)}| \leq 1$ for each $d \in D$. Then, as we have $z, \hat{z} \in W = \bar{C}^h(Z \cup A)$ such that $\mathsf{d}(z) = \mathsf{d}(\hat{z})$, we know that $\bar{Z} \equiv W \setminus A$ must be nonempty. Now, we have

$$\bar{C}^h(\bar{Z} \cup A) = \bar{C}^h((W \setminus A) \cup A) = \bar{C}^h(W \cup A) = \bar{C}^h((Z \cup A) \cup A) = \bar{C}^h(Z \cup A) = W \supseteq \bar{Z}, \tag{5}$$

where the third equality follows from the fact that $\bar{C}^h$ satisfies the irrelevance of rejected contracts condition and $W = \bar{C}^h(Z \cup A)$. Combining (5) with (4) (for the $d \in \mathsf{d}(\bar{Z}) \subseteq D$) shows that $\bar{Z}$ blocks A under $\bar{C}$, contradicting the stability of A under $\bar{C}$.

The preceding three observations show that A is stable with respect to C.

Proof of Theorem 1

Let $\bar{C}$ be a substitutable completion for C. By Theorem 3 of Hatfield and Kominers (2012), the (generalized) doctor-proposing cumulative offer process of Hatfield and Milgrom (2005) yields a (many-to-many) matching outcome A that is stable with respect to $\bar{C}$. By Lemma 1, A is stable with respect to C.

Proof of Theorem 2

We begin by proving a result which is slightly stronger than the first part of Theorem 2.

Theorem A.1. *If $\bar{C}$ is a substitutable completion of C, then the outcome of the doctor-proposing cumulative offer process under $\bar{C}$ is the same as the outcome of the doctor-proposing cumulative offer process under C.*

Proof. We fix a profile of choice functions $\bar{C}$ that substitutably completes C. We show by induction that the cumulative offer process under $\bar{C}$ corresponds step-by-step to the

cumulative offer process under C; it follows immediately that those processes then have the same outcome.

Let A^τ be the set of available contracts at the end of Step τ of the cumulative offer process under C; similarly, let $\bar{A}^\tau$ be the set of available contracts at the end of Step τ of the cumulative offer process under $\bar{C}$. Our inductive hypotheses are that

1. $A^\tau = \bar{A}^\tau$ and

2. at each Step τ, we have, for each $h \in H$, that $C^h(A^\tau) = \bar{C}^h(\bar{A}^\tau)$.

It follows immediately from the definition of the cumulative offer process that $A^1 = \bar{A}^1$. Moreover, since $A^1 = \bar{A}^1$ has at most one contract with each doctor, $C^h(A^1) = \bar{C}^h(A^1) = \bar{C}^h(\bar{A}^1)$ for all $h \in H$; therefore, the second inductive hypothesis is also satisfied at $\tau = 1$.

Hence, we suppose that $A^{\tau-1} = \bar{A}^{\tau-1}$ and suppose that for each $h \in H$, we have $C^h(A^{\tau-1}) = \bar{C}^h(\bar{A}^{\tau-1})$. By construction, then, the same set of doctors is held at the beginning of Step τ of both the cumulative offer process under C and the cumulative offer process under $\bar{C}$; hence, the same set of doctors makes proposals in Step τ of both processes. Moreover, since $A^{\tau-1} = \bar{A}^{\tau-1}$ (i.e., the same sets of contracts have been proposed prior to Step τ), we know that each doctor proposing in Step τ proposes the same contract in both cumulative offer processes. Consequently, we see immediately that $A^\tau = \bar{A}^\tau$.

Now, since $A^\tau = \bar{A}^\tau$, we have that $\bar{C}^h(\bar{A}^\tau) = \bar{C}^h(A^\tau)$. To prove the second inductive hypothesis, suppose that

$$\bar{C}^h(\bar{A}^\tau) = \bar{C}^h(A^\tau) \neq C^h(A^\tau), \tag{6}$$

seeking a contradiction. Since $\bar{C}^h$ completes C^h, there exists $z, \hat{z} \in \bar{C}^h(A^\tau)$ such that $\mathsf{d}(z) = \mathsf{d}(\hat{z})$. Now, we can not have $\{z, \hat{z}\} \subseteq \bar{C}^h(A^{\tau-1})$, as $C^h(A^{\tau-1})$ contains at most one contract with each doctor, and $\bar{C}^h(A^{\tau-1}) = \bar{C}^h(\bar{A}^{\tau-1}) = C^h(A^{\tau-1})$ by the second inductive hypothesis. Thus, without loss of generality, we have $z \notin \bar{C}^h(A^{\tau-1})$. But then, we have a contradiction to the substitutability of $\bar{C}^h$, as $z \notin \bar{C}^h(A^{\tau-1})$, but $z \in \bar{C}^h(A^\tau)$, and $A^{\tau-1} \subseteq A^\tau$. □

Theorem A.1 shows the first part of Theorem 2 does not depend on the existence of a substitutable completion that satisfies the irrelevance of rejected contracts condition. However, the irrelevance of rejected contracts condition is necessary for the second part of Theorem 2: If $\bar{C}$ is a substitutable completion of C that satisfies the irrelevance of rejected contracts condition, then Theorem 3 of Hatfield and Kominers (2012) implies that the outcome Y of the cumulative offer process under $\bar{C}$ is stable with respect to $\bar{C}$; Lemma 1 then implies that Y is stable under C, as well.

Proof of Theorem 3

Consider any substitutable completion $\bar{C}$ of C such that $\bar{C}^h$ satisfies the Law of Aggregate Demand (and the irrelevance of rejected contracts condition) for each $h \in H$. As the doctor-proposing deferred acceptance mechanism selects the doctor-optimal stable outcome under the completed choice profile $\bar{C}$, it follows from Theorem 10 of Hatfield and Kominers (2012) (which extends the result of Hatfield and Kojima (2009) to the setting of matching in networks) that the doctor-proposing deferred acceptance mechanism is (group) strategy-proof for doctors.[37]

B Preferences with Multiple Substitutable Completions

In this appendix, we show that different substitutable completions may lead to different sets of stable outcomes. As in Appendix A, we denote by $\mathsf{d}(x)$ the doctor associated with contract x.

Let $H = \{h\}$, $D = \{d, e\}$, and $X = \{x, \hat{x}, y, \hat{y}\}$ where $\mathsf{d}(x) = \mathsf{d}(\hat{x}) = d$ and $\mathsf{d}(y) = \mathsf{d}(\hat{y}) = e$. We consider the hospital choice function C^h induced by the preference relation

$$\succ_h\colon \{x, \hat{y}\} \succ \{\hat{x}, y\} \succ \{\hat{x}, \hat{y}\} \succ \{x, y\} \succ \{\hat{x}\} \succ \{\hat{y}\} \succ \{x\} \succ \{y\} \succ \varnothing,$$

[37] Formally, a mechanism (such as the cumulative offer process) is *group strategy-proof for doctors* if, for any choice function profile C and set of doctors $\tilde{D} \subseteq D$, there is no alternative choice function profile $(\tilde{C}^{\tilde{D}}, C^{D \setminus \tilde{D}}, C^H)$ such that every doctor in $\tilde{D}$ strictly prefers the outcome of the mechanism under $(\tilde{C}^{\tilde{D}}, C^{D \setminus \tilde{D}}, C^H)$ to the outcome of the mechanism under C.

along with choice functions C^d and C^e respectively induced by the preference relations

$$\succ_d : \{x\} \succ \{\hat{x}\} \succ \varnothing$$
$$\succ_e : \{y\} \succ \{\hat{y}\} \succ \varnothing.$$

There are three outcomes stable under C: $\{x, \hat{y}\}$, $\{\hat{x}, y\}$, and $\{x, y\}$.

Additionally, there are two different substitutable completions of C^h, induced respectively by the preference relations

$$\{y, \hat{y}\} \succ \{x, \hat{y}\} \succ \{\hat{x}, y\} \succ \{\hat{x}, \hat{y}\} \succ \{x, y\} \succ \{\hat{x}\} \succ \{\hat{y}\} \succ \{x\} \succ \{y\} \succ \varnothing \text{ and}$$
$$\{x, \hat{x}\} \succ \{x, \hat{y}\} \succ \{\hat{x}, y\} \succ \{\hat{x}, \hat{y}\} \succ \{x, y\} \succ \{\hat{x}\} \succ \{\hat{y}\} \succ \{x\} \succ \{y\} \succ \varnothing.$$

The completed choice profiles induced by these preference relations yield different sets of stable outcomes: $\{\hat{x}, y\}$ and $\{x, y\}$ are stable under the first, while $\{x, \hat{y}\}$ and $\{x, y\}$ are stable under the second.

C The Doctor-Proposing Deferred Acceptance Algorithm

The (doctor-proposing) deferred acceptance process under C proceeds as follows.

Step 1: Each doctor *proposes* his most-preferred contract from X under C (assuming there is one); the set of proposed contracts is denoted R^1. Each hospital h *holds* its favorite set of contracts from those that have been proposed, i.e., $C^h(R^1)$. Let the set of held contracts be denoted $G^1 \equiv \cup_{h \in H} C^h(R^1_h)$.

Step τ: Each doctor for whom no contract is currently held proposes his most-preferred contract that has not yet been proposed (if any), i.e., his most preferred contract from $X \setminus \cup_{\sigma=1}^{\tau-1} R^\sigma$ under C. If no contract is proposed, then the algorithm terminates and the outcome is the simply the set of held contracts, i.e., $G^{\tau-1}$. Otherwise, the set of contracts proposed in Step τ is denoted R^τ; each hospital h holds its favorite set of

contracts from those that have been proposed and those currently held by that hospital, i.e., $G^\tau \equiv \cup_{h \in H} C^h(R^\tau \cup G_h^{\tau-1})$; and the algorithm proceeds to Step $\tau + 1$.

D Bilaterally Substitutable Preferences

Hatfield and Kojima (2010) introduced the bilateral substitutability condition, which is weaker than substitutability but nevertheless sufficient to guarantee the existence of stable many-to-one matching with contracts outcomes. Here, we show that there exist substitutably completable choice functions that are not bilaterally substitutable.

First, we recall the formal statement of the bilateral substitutability condition. As in Appendix A, we denote by $\mathsf{d}(x)$ the doctor associated with contract x.

Definition. We say that the choice function C^i of $i \in D \cup H$ is *bilaterally substitutable* if for all $x, z \in X$ and $Y \subseteq X$ such that $\mathsf{d}(x), \mathsf{d}(z) \notin \mathsf{d}(Y)$, if $z \notin C^i(Y \cup \{z\})$, then $z \notin C^i(\{x\} \cup Y \cup \{z\})$.

All the substitutably completable choice functions we discussed in our main text examples are in fact bilaterally substitutable. However, there are substitutably completable choice functions that are not bilaterally substitutable. For example, let $H = \{h\}$, $D = \{d, e, f\}$, and $X = \{x, y, \hat{y}, z\}$ where $d = \mathsf{d}(x)$, $e = \mathsf{d}(y) = \mathsf{d}(\hat{y})$, and $f = \mathsf{d}(z)$. Consider the hospital preference relation

$$\succ_h\colon \{x, y, z\} \succ \{\hat{y}\} \succ \{x, y\} \succ \{x, z\} \succ \{y, z\} \succ \{y\} \succ \{x\} \succ \{z\} \succ \varnothing.$$

The preference relation $\succ_h$ induces a choice function C^h that is not bilaterally substitutable.[38]

Even though C^h is not bilaterally substitutable, it may be substitutably completed via the addition of preferences over $\{y, \hat{y}\}$: the choice function induced by the preference relation

$$\{y, \hat{y}\} \succ \{x, y, z\} \succ \{\hat{y}\} \succ \{x, y\} \succ \{x, z\} \succ \{y, z\} \succ \{y\} \succ \{x\} \succ \{z\} \succ \varnothing$$

is substitutable, satisfies the irrelevance of rejected contracts condition, and completes C^h.

[38] Note that $z \notin \{\hat{y}\} = C^h(\{y, \hat{y}, z\})$, but $z \in \{x, y, z\} = C^h(\{x, y, \hat{y}, z\})$, even though $\mathsf{d}(x), \mathsf{d}(z) \notin \mathsf{d}(\{y, \hat{y}\})$.

The preceding example demonstrates that bilateral substitutability does not imply substitutable completability. Thus, we see that substitutable completability is truly a "new" sufficient condition for the existence of stable many-to-one allocations—it includes a class of choice functions that were not previously known to have stable allocations guaranteed.

Interestingly, however, substitutable completability is not strictly weaker than bilateral substitutability. To see this, we consider a setting where $D = \{d, e\}$, $H = \{h\}$, and $X = \{x, y, \hat{x}, \hat{y}\}$, with $\mathsf{d}(x) = \mathsf{d}(\hat{x}) = d$ and $\mathsf{d}(y) = \mathsf{d}(\hat{y}) = e$. Consider the choice function C^h induced by the preference relation

$$\{x, y\} \succ \{\hat{x}\} \succ \{\hat{y}\} \succ \{x\} \succ \{y\} \succ \varnothing.$$

It is straightforward to check that C^h is bilaterally substitutable. But suppose that there were a substitutable completion $\bar{C}^h$ of C^h: We would need to have $\bar{C}^h(\{\hat{x}, y\}) = \{\hat{x}\}$ and $\bar{C}^h(\{\hat{x}, \hat{y}\}) = \{\hat{x}\}$, as $\bar{C}^h$ completes C^h; these facts imply that

$$\bar{C}^h(\{\hat{x}, y, \hat{y}\}) = \{\hat{x}\}, \tag{7}$$

as $\bar{C}^h$ is substitutable. As $\bar{C}^h$ completes C^h, we would also need to have $\bar{C}^h(\{x, \hat{y}\}) = \{\hat{y}\}$; this fact, along with (7), would imply that

$$\bar{C}^h(\{x, \hat{x}, y, \hat{y}\}) = \{\hat{x}\},$$

as $\bar{C}^h$ is substitutable. But then $\bar{C}^h$ could not be a completion—a contradiction—as $C^h(\{x, \hat{x}, y, \hat{y}\}) = \{x, y\} \neq \{\hat{x}\} = \bar{C}^h(\{x, \hat{x}, y, \hat{y}\})$ and $\bar{C}^h(\{x, \hat{x}, y, \hat{y}\}) = \{\hat{x}\}$ does not contain two contracts with the same doctor.

Figure 1 shows the relationship between substitutable completability and the substitutability structures introduced in this prior literature (assuming the irrelevance of rejected contracts condition).

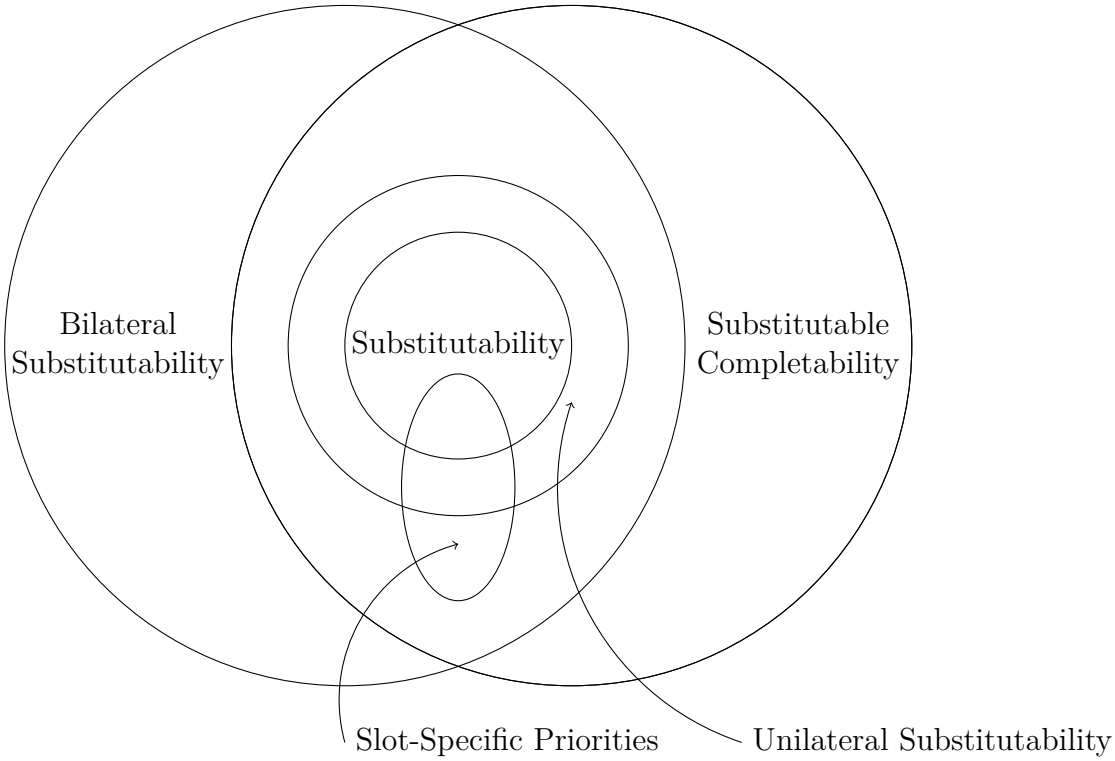

Figure 1: The relationship between substitutability concepts for many-to-one matching with contracts.

E The Structure of the Set of Stable Outcomes under Substitutable Completability

Like most arguments for strategy-proofness, Theorem 3 uses a form of the *rural hospitals theorem* (Roth, 1984; Hatfield and Milgrom, 2005), which states that, under choice functions that are substitutable and satisfy the Law of Aggregate Demand (and the irrelevance of rejected contracts condition), the number of contracts each agent signs is invariant across stable outcomes. However, while the rural hospitals theorem applies for any fixed substitutable completion that satisfies the Law of Aggregate Demand, its conclusion may not hold under the original choice profile C.[39] For instance, in our running Sherlock–Watson example, there are two stable outcomes with different numbers of contracts, even though C^h satisfies the Law of Aggregate Demand itself and has a substitutable completion that satisfies the Law of Aggregate Demand (and the irrelevance of rejected contracts condition).

Theorem 2 shows that when hospitals' choice functions are substitutably completable, the doctor-proposing cumulative offer process finds a stable outcome. However, that outcome, which we denote Y, may not be doctor-optimal among outcomes stable under the original choice profile C; in fact, there may not exist a doctor-optimal stable outcome under the original choice profile. For instance, in our Sherlock–Watson example, Sherlock prefers the stable outcome $\{\mathsf{S}^c\}$, while Watson prefers the other stable outcome $\{\mathsf{S}^r, \mathsf{W}^c\}$.[40] Nevertheless, for any completion $\bar{C}$ of C, the outcome Y *is* the doctor-optimal stable outcome under $\bar{C}$, in the sense that every doctor weakly prefers Y to every other outcome stable under $\bar{C}$; *this* fact implies that, when hospitals' choice functions have substitutable completions that also satisfy

[39] However, Theorem 2 implies that the rural hospitals theorem holds across completions, in the sense that the number of contracts each agent signs is invariant across outcomes that are stable under some substitutable completion that satisfies the Law of Aggregate Demand (and the irrelevance of rejected contracts condition).

[40] The Sherlock–Watson example also shows that the set of stable outcomes under a substitutably completable choice function need not form a lattice in the usual way, as Sherlock's and Watson's preferences over stable outcomes are not aligned. However, under the completion $\bar{C}$ with $\bar{C}^{\mathsf{h}}$ as defined in (1), only $\{\mathsf{S}^c\}$ is stable. Substitutable completion in a certain sense restores the lattice structure of stable outcomes observed by Hatfield and Milgrom (2005). Specifically, for any substitutable completion $\bar{C}$ of C (that satisfies the irrelevance of rejected contracts condition), we obtain a lattice of outcomes stable with respect to $\bar{C}$—which may not contain all the outcomes stable with respect to C.

the Law of Aggregate Demand (and the irrelevance of rejected contracts condition), using the doctor-proposing cumulative offer process incentivizes doctors to reveal their preferences truthfully (Theorem 3).

Jagadeesan (2016b) has recently identified a refinement of substitutable completability, *substitutable strict completability*, that fully restores the classical results on the structure of the set of stable outcomes.

F Tasks-and-Slots Priority Stuctures

We now describe a class of *tasks-and-slots priority structures* that includes both the gymnasium priority structures introduced in Section 4 (see Appendix F.1) and the *slot-specific priority structures* of Kominers and Sönmez (2014) (see Appendix F.3). Tasks-and-slots priority structures include two different types of positions: *tasks* and *slots*. "Task" positions are always filled before "slot" positions. The order in which tasks are filled may depend on the set of contracts available; however, any two tasks either have identical priority orderings or find disjoint sets of contracts acceptable. Meanwhile, in principle, any contract can be accepted by any slot, but the sequence in which slots are filled can not depend on the set of contracts available.

As in Appendix A, we denote by $\mathsf{d}(x)$ the doctor associated with contract x; similarly, we denote by $\mathsf{d}(Y)$ the set of doctors associated with some contract in Y, i.e., $\mathsf{d}(Y) = \cup_{y \in Y} \mathsf{d}(y)$.

For each hospital h, there is a set of *slots* $\mathcal{S}^h$ and a (disjoint) set of *tasks* $\mathcal{T}^h$; the set of *positions* $\mathcal{P}^h$ is the union of slots and tasks, i.e., $\mathcal{P}^h \equiv \mathcal{S}^h \cup \mathcal{T}^h$. For each slot $s \in \mathcal{S}^h$, there exists a *priority ordering* $\succ_s$ over elements of X and an *outside option* $\emptyset$. Similarly, for each task $t \in \mathcal{T}^h$, there exists a *priority ordering* $\succ_t$ over elements of X and an *outside option* $\emptyset$. However, the set of tasks can be partitioned into a set of classes $\mathscr{C}$ where, for any two tasks $t, \bar{t}$ in the same class $\mathcal{C} \in \mathscr{C}$, the tasks have identical priority orderings, i.e., $\succ_t = \succ_{\bar{t}}$, while tasks in distinct classes find disjoint sets of contracts acceptable.[41] Finally, each hospital h

[41] We say that a given contract x is *acceptable* for a given position p if it is preferred to the null contract, i.e., $x \succ_p \emptyset$.

is also endowed with a *precedence ordering* $\rhd_h^Y$ over positions in $\mathcal{P}^h$ that determines, as a function of the set of proposed contracts Y, the order in which positions will be filled.

We impose the following restrictions on the precedence ordering $\rhd_h$:

1. Tasks are filled before slots; that is, for all $Y \subseteq X$, for any task $t \in \mathcal{T}^h$ and any slot $s \in \mathcal{S}^h$, we have that $t \rhd_h^Y s$.

2. Slots are filled in the same order regardless of the set of contracts available; that is, for all $Y, \bar{Y} \subseteq X$, for any slots $s, \bar{s} \in \mathcal{S}^h$, if $s \rhd_h^Y \bar{s}$ then $s \rhd_h^{\bar{Y}} \bar{s}$.

Finally, the hospital has a *quota* q^h of positions it wishes to fill; we assume that $q^h \geq |\mathcal{T}^h|$. A *tasks-and-slots priority structure* is a tuple $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, q^h)$.

If the choice function C^h of hospital h is *induced by the tasks-and-slots priority structure* $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, q^h)$, then we compute $C^h(Y)$ for any set of available contracts Y as follows:

1. Initialize the set of *available contracts* as $A^0 = Y$ and the set of *selected contracts* as $G^0 = \varnothing$.

2. Label the positions in $\mathcal{P}^h$ as $p^1, p^2, \ldots, p^{|\mathcal{P}^h|}$, where p^ℓ is the ℓ^{th} highest position according to the precedence order $\rhd_h^Y$.

3. If the number of held contracts is equal to the quota, i.e., $|G^{\ell-1}| = q^h$, or if all the positions have been considered, i.e., $\ell = |\mathcal{P}^h| + 1$, continue to Step 4. Otherwise, let x^ℓ be the $\succ_{p^\ell}$-maximal contract in $A^{\ell-1} \cup \{\emptyset\}$. If $x^\ell \neq \emptyset$, then:

   (a) add x^ℓ to the set of selected contracts, i.e., let $G^\ell \equiv G^{\ell-1} \cup \{x^\ell\}$; and

   (b) remove any contracts associated with $\mathsf{d}(x^\ell)$ from the set of available contracts, i.e., let $A^\ell \equiv A^{\ell-1} \setminus Y_{\mathsf{d}(x^\ell)}$.

   If instead $x^\ell = \emptyset$, let $G^\ell = G^{\ell-1}$ and $A^\ell = A^{\ell-1}$. Increment ℓ and return to Step 3.

4. Finally, take the choice of h from Y to be the set of selected contracts, i.e., set $C^h(Y) = G^{\ell-1}$.

As constructed, a choice function induced by a tasks-and-slots priority structure does not necessarily satisfy the irrelevance of rejected contracts condition, as we demonstrate in Appendix F.2.

We now show the main result of this appendix.

Theorem F.1. *Any choice function induced by a tasks-and-slots priority structure has a substitutable completion that satisfies the Law of Aggregate Demand and the irrelevance of rejected contracts condition.*

We suppose that the choice function C^h is induced by the tasks-and-slots priority structure $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, q^h)$. We construct a completion $\bar{C}^h$ of C^h by relaxing the constraint that the hospital can choose at most one contract with each doctor. That is, under $\bar{C}^h$, when a contract x is chosen, we remove only the contract x from consideration for other positions, instead of removing all the contracts with the doctor $\mathsf{d}(x)$. More formally, $\bar{C}^h$ is the *completion induced by the tasks-and-slots priority structure* $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, q^h)$, and is generated by the following algorithm:

1. Initialize the set of *available contracts* as $\bar{A}^0 = Y$ and the set of *selected contracts* as $\bar{G}^0 = \varnothing$.

2. Label the positions in $\mathcal{P}^h$ as $p^1, p^2, \ldots, p^{|\mathcal{P}^h|}$, where p^ℓ is the ℓ^{th} highest position according to the precedence order $\rhd_h^Y$.

3. If the number of held contracts is equal to the quota, i.e., $|\bar{G}^{\ell-1}| = q^h$, or if all the positions have been considered, i.e., $\ell = |\mathcal{P}^h| + 1$, continue to Step 4. Otherwise, let x^ℓ be the $\succ_{p^\ell}$-maximal contract in $\bar{A}^{\ell-1} \cup \{\emptyset\}$. If $\bar{x}^\ell \neq \emptyset$, then:

   (a) add x^ℓ to the set of selected contracts, i.e., let $\bar{G}^\ell \equiv \bar{G}^{\ell-1} \cup \{\bar{x}^\ell\}$; and

   (b) remove $\bar{x}^\ell$ from the set of available contracts, i.e., let $\bar{A}^\ell \equiv \bar{A}^{\ell-1} \setminus \{\bar{x}^\ell\}$.

If instead $\bar{x}^\ell = \emptyset$, let $\bar{G}^\ell = \bar{G}^{\ell-1}$ and $\bar{A}^\ell = \bar{A}^{\ell-1}$. Increment ℓ and return to Step 3.

4. Finally, take the choice of h from Y to be the set of selected contracts, i.e., set $C^h(Y) = \bar{G}^{\ell-1}$.

Note that $\bar{C}^h$ is defined using the same algorithm as C^h except that in Step 3b of the computation of $\bar{C}^h(Y)$, we remove just $\{\bar{x}^\ell\}$ from consideration for lower-precedence positions, while in Step 3b of the computation of $C^h(Y)$, we remove $Y_{\mathsf{d}(x^\ell)} \supseteq \{x^\ell\}$ from consideration for lower-precedence positions.

Claim 1. *The choice function $\bar{C}^h$ completes C^h.*

Proof. It suffices to show that for each $Y \subseteq X$, if $\bar{C}^h(Y) \neq C^h(Y)$, then there is some doctor $d \in D$ such that $\bar{C}^h(Y)$ contains two contracts associated with d.

If $\bar{C}^h(Y) \neq C^h(Y)$, then there is some first instance for which $x^\ell \neq \bar{x}^\ell$, i.e., some minimal ℓ such that $x^\ell \neq \bar{x}^\ell$. Now, the only difference between the algorithm defining C^h and that defining $\bar{C}^h$ arises in Step 3b: in computing $C^h(Y)$, for each $m < \ell$, we set $A^m = A^{m-1} \setminus Y_{\mathsf{d}(x^m)}$, whereas in computing $\bar{C}^h(Y)$, we set $\bar{A}^m = \bar{A}^{m-1} \setminus \{\bar{x}^m\}$. Thus, since $x^m = \bar{x}^m$ for all $m \leq \ell$ by construction, we see that $A^{\ell-1}$, the set of contracts available to be assigned to h in iteration ℓ of Step 3 of the computation of $C^h(Y)$, differs from $\bar{A}^{\ell-1}$ (the set of contracts available to be assigned to h in iteration ℓ of Step 3 of the computation of $\bar{C}^h(Y)$) only in that additional contracts with doctors in $\mathsf{d}(G^{\ell-1})$ are available; specifically, $\bar{A}^{\ell-1} = A^{\ell-1} \cup (Y_{\mathsf{d}(G^{\ell-1})} \setminus G^{\ell-1})$.

Now, the contract $\bar{x}^\ell$ selected in iteration ℓ of Step 3 of the computation of $\bar{C}^h(Y)$ differs from x^ℓ, the contract selected in iteration ℓ of Step 3 of the computation of $C^h(Y)$. Moreover, $\bar{x}^\ell$ is maximal among contracts in the set $\bar{A}^{\ell-1}$ of contracts available to be assigned in iteration ℓ of the computation of $\bar{C}^h(Y)$. Thus, we have that $\bar{x}^\ell \in \bar{A}^{\ell-1} \setminus A^{\ell-1} = Y_{\mathsf{d}(G^{\ell-1})} \setminus G^{\ell-1}$; so, in particular, $\mathsf{d}(\bar{x}^\ell) \in \mathsf{d}(G^{\ell-1})$. Hence, when computing $\bar{C}^h(Y)$, we have that $\bar{G}^m$ contains at least two contracts associated with the doctor $\mathsf{d}(\bar{x}^\ell)$ for all $m \geq \ell$. Hence, $\bar{C}^h(Y)$ contains at least two contracts associated with the doctor $\mathsf{d}(\bar{x}^\ell)$. □

Claim 2. *The completion $\bar{C}^h$ induced by the tasks-and-slots priority structure*

$$(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, q^h)$$

is equivalent to the completion induced by the tasks-and-slots priority structure

$$(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \blacktriangleright_h, q^h),$$

where $\blacktriangleright_h^Y = \rhd_h^\varnothing$ for all $Y \subseteq X$.

Proof. Let m denote the total number of tasks, i.e., $m = |\mathcal{T}^h|$. Recall that all tasks in a given class $\mathcal{C}$ use the same priority ordering; we abuse notation slightly by denoting that priority ordering $\succ_{\mathcal{C}}$. Let

$$M_{\mathcal{C}} \equiv \{x \in Y : x \succ_{\mathcal{C}} \emptyset \text{ and } x \text{ is one of the } |\mathcal{C}| \text{ highest-ranked elements of } Y \text{ according to } \succ_{\mathcal{C}}\}.$$

Now, *for any precedence order*, as any two tasks in different classes find disjoint sets of contracts acceptable, and any two tasks in the same class agree on the priority ordering over contracts, we compute that $\bar{G}^m = \cup_{\mathcal{C}\in\mathscr{C}} M_{\mathcal{C}}$. It then follows that, again for any precedence ordering, the set of available contracts at the end of iteration m of the computation $\bar{C}^h(Y)$ is exactly $Y \setminus \bar{G}^m$.

Moreover, for every precedence order, slots are filled only after tasks are considered, and slots are always filled in the same order. Hence, as for any precedence order the set of contracts available to be assigned to slots is always $Y \setminus \bar{G}^m$, the set of contracts assigned to slots $(\bar{G}^{q^h} \setminus \bar{G}^m)$ is independent of the precedence order.

It follows that the set of contracts chosen by the completion $\bar{C}^h$ induced by the tasks-and-slots priority structure $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, q^h)$ is the same as the set of contracts chosen by the completion induced by the tasks-and-slots priority structure $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \blacktriangleright_h, q^h)$. □

Given Claim 2, it is without loss of generality to assume that $\rhd_h^Y = \rhd_h^\varnothing$ for all $Y \subseteq X$, i.e., that $\rhd_h$ is a fixed precedence order. Accordingly, we shall drop the superscript on $\rhd_h$ for the remainder of the proof.

Claim 3. *The completion $\bar{C}^h$ induced by the tasks-and-slots priority structure*

$$(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, \bar{q}^h)$$

is substitutable and satisfies the Law of Aggregate Demand.

Proof. For any set of contracts Y, we let $\bar{A}^\ell_Y$ denote the set of contracts available to be assigned to positions after iteration ℓ of Step 3 of the computation of $\bar{C}^h(Y)$. Analogously, we let $\bar{G}^\ell_Y$ denote the set of contracts selected by the end of iteration ℓ of Step 3 of the computation of $\bar{C}^h(Y)$.

To show that $\bar{C}^h$ is substitutable, we show that for any $z \in X_h$ and $Y \subseteq \hat{Y} \subseteq X_h$,

$$\text{if } z \notin \bar{C}^h(Y) \text{ but } z \in Y, \text{ then } z \notin \bar{C}^h(\hat{Y}). \tag{8}$$

To show that $\bar{C}^h$ satisfies the Law of Aggregate Demand, we show that for any $Y \subseteq \hat{Y} \subseteq X_h$, we have that

$$|\bar{C}(Y)| \leq |\bar{C}(\hat{Y})|. \tag{9}$$

We show both (8) and (9) show by way of the following claim:

Subclaim 1. *At each iteration ℓ of Step 3 of the computations of $\bar{C}^h(\hat{Y})$ and $\bar{C}^h(Y)$, we have $\bar{A}^\ell_{\hat{Y}} \supseteq \bar{A}^\ell_Y$ and $|\bar{G}^\ell_{\hat{Y}}| \geq |\bar{G}^\ell_Y|$.*

Proof. We proceed by induction. First, we note that for $\ell = 0$, we have $\bar{A}^0_{\hat{Y}} = \hat{Y} \supseteq Y = \bar{A}^0_Y$ and $\bar{G}^0_{\hat{Y}} = \varnothing = \bar{G}^0_Y$, so we assume that the claim holds for all $m < \ell$. At iteration $\ell > 0$, let p be the ℓ^{th} highest position according to the precedence ordering $\rhd_h$, and let x^ℓ_Y be the $\succ_p$-maximal contract in $\bar{A}^\ell_Y$ and $x^\ell_{\hat{Y}}$ be the $\succ_p$-maximal contract in $\bar{A}^\ell_{\hat{Y}}$. There are four possibilities:

Case 1: $x^\ell_{\hat{Y}} = x^\ell_Y \neq \emptyset$**.** In this case, $\bar{A}^\ell_{\hat{Y}} = \bar{A}^{\ell-1}_{\hat{Y}} \setminus \{x^\ell_Y\}$ and $\bar{A}^{\ell-1}_Y = \bar{A}^\ell_Y \setminus \{x^\ell_Y\}$. Since by the inductive hypothesis we have $\bar{A}^{\ell-1}_{\hat{Y}} \supseteq \bar{A}^{\ell-1}_Y$, it immediately follows that $\bar{A}^\ell_{\hat{Y}} \supseteq \bar{A}^\ell_Y$. Moreover, since by the inductive hypothesis we have $|\bar{G}^{\ell-1}_{\hat{Y}}| \geq |\bar{G}^{\ell-1}_Y|$, we know that $|\bar{G}^\ell_{\hat{Y}}| = |\bar{G}^{\ell-1}_{\hat{Y}}| + 1 \geq |\bar{G}^{\ell-1}_Y| + 1 = |\bar{G}^\ell_Y|$.

Case 2: $x^\ell_{\hat{Y}} = x^\ell_Y = \emptyset$**.** In this case, $\bar{A}^\ell_{\hat{Y}} = \bar{A}^{\ell-1}_{\hat{Y}}$ and $\bar{A}^\ell_Y = \bar{A}^{\ell-1}_Y$; moreover, $\bar{G}^\ell_{\hat{Y}} = \bar{G}^{\ell-1}_{\hat{Y}}$ and $\bar{G}^\ell_Y = \bar{G}^{\ell-1}_Y$. As by the inductive hypothesis we have $\bar{A}^{\ell-1}_{\hat{Y}} \supseteq \bar{A}^{\ell-1}_Y$, it immediately follows that $\bar{A}^\ell_{\hat{Y}} \supseteq \bar{A}^\ell_Y$. Moreover, since by the inductive hypothesis we have $|\bar{G}^{\ell-1}_{\hat{Y}}| \geq |\bar{G}^{\ell-1}_Y|$, we know that $|\bar{G}^\ell_{\hat{Y}}| = |\bar{G}^{\ell-1}_{\hat{Y}}| \geq |\bar{G}^{\ell-1}_Y| = |\bar{G}^\ell_Y|$.

Case 3: $x^\ell_{\hat{Y}} \neq x^\ell_Y$ **and** $x^\ell_Y = \emptyset$**.** In this case, note that $x^\ell_{\hat{Y}} \neq \emptyset$ implies that $x^\ell_{\hat{Y}} \succ_p \emptyset$. This implies that $x^\ell_{\hat{Y}} \notin \bar{A}^{\ell-1}_Y$, as otherwise we would not have $x^\ell_Y = \emptyset$. Since by the inductive hypothesis we have $\bar{A}^{\ell-1}_{\hat{Y}} \supseteq \bar{A}^{\ell-1}_Y$, it immediately follows that $\bar{A}^\ell_{\hat{Y}} = \bar{A}^{\ell-1}_{\hat{Y}} \setminus \{x^\ell_{\hat{Y}}\} \supseteq \bar{A}^{\ell-1}_Y = \bar{A}^\ell_Y$. Moreover, since by the inductive hypothesis we have $|\bar{G}^{\ell-1}_{\hat{Y}}| \geq |\bar{G}^{\ell-1}_Y|$, we know that $|\bar{G}^\ell_{\hat{Y}}| = |\bar{G}^{\ell-1}_{\hat{Y}}| + 1 \geq |\bar{G}^{\ell-1}_Y| = |\bar{G}^\ell_Y|$.

Case 4: $x^\ell_{\hat{Y}} \neq x^\ell_Y$ **and** $x^\ell_Y \neq \emptyset$**.** First, we note that $x^\ell_{\hat{Y}} \succ_p x^\ell_Y$, as by the inductive hypothesis we have $\bar{A}^{\ell-1}_{\hat{Y}} \supseteq \bar{A}^{\ell-1}_Y$ and p^ℓ is assigned the $\succ_p$-maximal contract in Step 3.[42] Hence, we must have $x^\ell_{\hat{Y}} \notin \bar{A}^{\ell-1}_Y$, as otherwise $x^\ell_Y \neq x^\ell_{\hat{Y}}$ would not be selected in the ℓ^{th} iteration of Step 3 of the compuation of $\bar{C}^h(Y)$. Since by the inductive hypothesis we have $\bar{A}^{\ell-1}_{\hat{Y}} \supseteq \bar{A}^{\ell-1}_Y$, it immediately follows that $\bar{A}^\ell_{\hat{Y}} = \bar{A}^{\ell-1}_{\hat{Y}} \setminus \{x^\ell_{\hat{Y}}\} \supseteq \bar{A}^{\ell-1}_Y \setminus \{x^\ell_Y\} = \bar{A}^\ell_Y$. Moreover, since by the inductive hypothesis we have $|\bar{G}^{\ell-1}_{\hat{Y}}| \geq |\bar{G}^{\ell-1}_Y|$, we know that $|\bar{G}^\ell_{\hat{Y}}| = |\bar{G}^{\ell-1}_{\hat{Y}}| + 1 \geq |\bar{G}^{\ell-1}_Y| + 1 = |\bar{G}^\ell_Y|$. □

Subclaim 1 implies the substitutability of $\bar{C}^h$ (that is, (8)), as: For each iteration ℓ of Step 3, the ℓ^{th} highest-precedence position p^ℓ is assigned the $\succ_{p^\ell}$-maximal contract from the set of contracts still available. Thus, if $z \notin \bar{C}^h(Y)$, then z is not selected in any iteration of Step 3 the computation of $\bar{C}^h(Y)$, so it must be that z is not the $\succ_{p^\ell}$-maximal element of $\bar{A}^{\ell-1}_Y \cup \{\emptyset\}$ for any ℓ reached in the computation of $\bar{C}^h(Y)$. But then, as $\bar{A}^{\ell-1}_{\hat{Y}} \cup \{\emptyset\} \supseteq \bar{A}^{\ell-1}_Y \cup \{\emptyset\}$ (by Claim 1), we see that z can not be the $\succ_{p^\ell}$-maximal element of $\bar{A}^{\ell-1}_{\hat{Y}} \cup \{\emptyset\}$ for any ℓ reached in the computation of $\bar{C}^h(Y)$. Moreover, we have (again by Claim 1) that $|\bar{G}^\ell_{\hat{Y}}| \geq |\bar{G}^\ell_Y|$; hence if the computation of $\bar{C}^h(Y)$ stops at iteration ℓ of Step 3, then the computation of $\bar{C}^h(\hat{Y})$ must stop at iteration $\hat{\ell} \leq \ell$. Thus, we see that z can not be selected

[42] In particular, this implies that $x^\ell_{\hat{Y}} \neq \emptyset$.

in the computation of $\bar{C}^h(\hat{Y})$.

Subclaim 1 also implies that $\bar{C}^h$ satisfies the Law of Aggregate Demand (that is, (9)), as: For each iteration ℓ of Step 3, we have that $|\bar{G}^\ell_{\hat{Y}}| \geq |\bar{G}^\ell_Y|$, so at any iteration ℓ of Step 3 before the quota is met, more contracts are assigned in the computation of $\bar{C}^h(\hat{Y})$ than in the computation of $\bar{C}^h(Y)$. Thus, if the computation of $\bar{C}^h(\hat{Y})$ ends at iteration $|\mathcal{P}^h| + 1$ of Step 3 (and, hence, the computation of $\bar{C}^h(Y)$ also ends at iteration $|\mathcal{P}^h| + 1$), we have that $|\bar{C}^h(\hat{Y})| = |\bar{G}^{|\mathcal{P}^h|}_{\hat{Y}}| \geq |\bar{G}^{|\mathcal{P}^h|}_Y| = |\bar{C}^h(Y)|$. Moreover, the computation of $\bar{C}^h(\hat{Y})$ ends at iteration $\ell < |\mathcal{P}^h| + 1$ of Step 3 only if $|\bar{G}^{\ell-1}_{\hat{Y}}| = q^h$. But in this case, the result is immediate, as $|\bar{C}^h(W)| \leq q^h = |\bar{G}^{\ell-1}_{\hat{Y}}| = \bar{C}^h(\hat{Y})$ for all $W \subseteq X$ (and, in particular, when $W = Y$). □

Claim 4. *The completion $\bar{C}^h$ induced by the tasks-and-slots priority structure*

$$(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, \bar{q}^h)$$

satisfies the irrelevance of rejected contracts condition.

Proof. Proposition 1 of Aygün and Sönmez (2012) shows that any substitutable choice function that satisfies the Law of Aggregate Demand also satisfies the irrelevance of rejected contracts condition.[43] Thus, our claim here follows directly from Claim 3. □

Taken together, the claims of this section show Theorem F.1.

F.1 Gymnasium Priority Structures

In this section, we show how every choice function induced by a gymnasium priority structure corresponds to a choice function induced by a tasks-and-slots priority structure.

Consider a gymnasium priority structure $(P^h, \{\succ_{(h,t)}\}_{t\in T}, \succ_{(h,\star)}, \rhd_h, q^h)$. We construct an associated tasks-and-slots priority structure $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \blacktriangleright_h, \bar{q}^h)$ as follows:

1. We take $\mathcal{T}^h = P^h$, that is, the set of tasks $\mathcal{T}^h$ corresponds to the set of subject-specific positions P^h. For each subject $t \in T$, we let $\mathcal{C}_t = \{p \in P^h : \mathtt{t}(p) = t\}$ be the set of tasks

[43] Although Aygün and Sönmez consider a many-to-one matching with contracts setting, their proof extends without change to the many-to-many matching with contracts setting.

corresponding to subject-specific positions associated with the subject t; note that, by definition, any two tasks in the same class have identical preference orderings and any two tasks in different classes have disjoint sets of acceptable contracts.

2. Next, we let $\mathcal{S}^h$ be such that $|\mathcal{S}^h| = q^h$ and let each slot $s \in \mathcal{S}^h$ have a priority ordering $\succ_s$ equal to $\succ_{(h,\star)}$.

3. We let $\blacktriangleright_h$ be such that it corresponds to $\rhd_h$ over the set $\mathcal{T}^h = P^h$ of tasks, and let $\blacktriangleright_h$ rank slots arbitrarily (but after all tasks).

4. Finally, we let $\bar{q}^h = q^h$.

It is clear that the choice function induced by the tasks-and-slots priority structure $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \blacktriangleright_h, \bar{q}^h)$ exactly corresponds to that induced by the gymnasium priority structure $(P^h, \{\succ_{(h,t)}\}_{t\in T}, \succ_{(h,\star)}, \rhd_h, q^h)$. This correspondence, combined with Theorem F.1, immediately yields Proposition 1.

F.2 A Choice Function Induced by a Tasks-and-Slots Priority Structure That Does Not Satisfy the Irrelevance of Rejected Contracts Condition

Under a tasks-and-slots priority structure, precedence orders can depend arbitrarily on the set of contracts available: in particular, they can depend on contracts which are unacceptable. Thus, the irrelevance of rejected contracts condition can naturally be violated, as the presence of an unacceptable contract can change the precedence order in a way that changes the set of contracts chosen.

For a simple example, let $D = \{d, e\}$, $H = \{h\}$, and $T = \{c, r\}$ where the contractual term c denotes working as a clinician and the contractual term r denotes working as a researcher. The set of contracts is given by $X = D \times \{h\} \times \{c, r\}$.

Hospital h has two positions, a clinician task and a researcher task, denoted $\mathcal{T}^h = \{c, r\}$;

the set of slots $\mathcal{S}^h$ is empty. The priority orderings for the tasks are:

$$\succ_c : (d, h, c) \succ \emptyset$$

$$\succ_r : (d, h, r) \succ \emptyset$$

and the precedence order is

$$\rhd_h^Y = \begin{cases} r \rhd c & e \in \mathsf{d}(Y) \\ c \rhd r & \text{otherwise.} \end{cases}$$

The choice function C^h induced by this tasks-and-slots priority structure does not satisfy the irrelevance of rejected contracts condition, as we have that $C^h(\{(d, h, c), (d, h, r)\}) = \{(d, h, c)\}$, while $C^h(\{(d, h, c), (d, h, r), (e, h, r)\}) = \{(d, h, r)\}$.

F.3 Applications

The tasks-and-slots priority framework generalizes the slot-specific priority framework of Kominers and Sönmez (2014).[44] Hence, the tasks-and-slots priority framework encompasses all of the slot-specific priority framework applications, including cadet–branch matching (Sönmez and Switzer, 2013; Sönmez, 2013), airline upgrade allocation (Kominers and Sönmez, 2015), and the design of affirmative action mechanisms (Kominers and Sönmez, 2015); our work here shows that the cumulative offer mechanism is stable and strategy-proof in all of those settings.

Building on our approach, Kojima et al. (2016) have shown a second way of obtaining the results of Sönmez and Switzer (2013): they consider the cadet–branch matching setting as a many-to-many matching model, and show that in that setting the branches' choice functions can be represented by $M^\natural$-concave functions. They then apply their Corollary 1 to show that the branches' "many-to-many" (or, equivalently, completed) choice functions are substitutable, satisfy the Law of Aggregate Demand, and satisfy the irrelevance of rejected contracts condition; hence, the cumulative offer mechanism is stable and strategy-proof (see also Kamada and Kojima (2012, 2014, 2015)).

[44] The slot-specific priority framework is recovered by setting the set of tasks to be empty.

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