Contract Design and Stability in Matching Markets

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Abstract

We develop a model of many-to-many matching with contracts which subsumes as special cases many-to-many matching markets and buyer-seller markets with heterogeneous and indivisible goods. In our setting, in contrast to results for the setting of many-to-one matching with contracts, preference substitutability is sufficient and necessary to guarantee the existence of a nonempty lattice of stable allocations. Several fundamental structural results, such as the rural hospitals theorem, extend to our setting. We apply these results to derive a novel condition on preferences, weaker than substitutability, that is sufficient for the existence of a lattice of stable allocations in the context of many-to-one matching with contracts; extensions of this condition yield rural hospitals and strategy-proofness results. We also develop a theory of language for many-to-many matching markets with contracts, and show a natural tradeoff between expressiveness and stability: bundling makes a contractual language less expressive, but encourages substitutability of agents' preferences over contracts.

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1 Introduction

We develop a model of many-to-many matching with contracts in which agents on two opposing sides of a market negotiate over contractual relationships, possibly signing multiple contracts. This setting models several real-world matching markets, such as the United Kingdom Medical Intern match (see Roth and Sotomayor (1990)) and the market used to allocate blood from blood banks to hospitals (see Jaume et al. (2009)). One important special case of our model is matching with couples, in which pairs of individuals may choose to act as a single agent which receives (at most) two assignments (see Klaus and Klijn (2005); Klaus et al. (2007).[1]

Our model includes many-to-one matching with contracts (Kelso and Crawford (1982),Hatfield and Milgrom (2005)) and many-to-many matching (Sotomayor (1999, 2004), Echenique and Oviedo (2006), Konishi and Ünver (2006)), and buyer-seller markets with heterogeneous and indivisible goods as special cases. It is substantively different from the only previous model many-to-many matching with contracts—that of Klaus and Walzl (2009)—as we allow a pair of agents to sign multiple contracts with each other simultaneously. (This distinction is material to our results, as we discuss in Section 6.1.2.)

We show that, in the setting of many-to-many matching with contracts, the domain of *substitutable preferences* is the maximal domain for which the existence of stable allocations can be guaranteed.[2] Our maximal domain result is particularly surprising because no analogous result holds in the Hatfield and Milgrom (2005) model of many-to-one matching with contracts (Hatfield and Kojima (2008, 2010)).

In addition to our maximal domain result, this paper makes several technical and

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[1] For example, in the United States National Resident Matching Program (NRMP), doctors may apply to the NRMP as a couple, submitting a preference list over pairs of job assignments and being assigned to two jobs (see Roth and Peranson (1999)).

[2] This means that substitutability of preferences is sufficient to guarantee the existence of stable allocations, and is necessary in the sense that if any one agent has preferences which are not subsitutable, then substitutable preferences can be found for other agents such that no stable allocation exists.

conceptual contributions: structural results for many-to-many stable allocations, a new condition on preferences that guarantees the existence of stable allocations in the setting of many-to-one matching with contracts, alternative characterizations of substitutability, and a theory of contract design.

1.1 Structural Results

After proving our maximal domain result, we study the structure of the set of stable allocations in the presence of both substitutable preferences and the *law of aggregate demand*, which states that an agent chooses (weakly) more contracts when the set of contracts available to that agent expands. Together, these conditions are sufficient to guarantee an extension of the *rural hospitals theorem* of Roth (1984a): each agent signs the same number of contracts at every stable allocation.[3] We also demonstrate that the stable allocation correspondence in our framework is Nash implementable whenever it is nonempty and there are at least three agents.[4] Finally, we show that when hospital preferences satisfy a more-stringent condition than substitutability (so-called *strong substitutability*) and doctor preferences are substitutable, stability is equivalent to a more stringent solution concept: *strong stability*.[5]

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[3] The *rural hospitals theorem* of Roth (1984a) shows that hospitals which fail to fill their positions in some stable matching are allocated the same number of doctors in every stable matching. Our result generalizes an analogous result of Hatfield and Milgrom (2005) for many-to-one matching with contracts.

[4] This implementability result subsumes those of Kara and Sönmez (1996, 1997) and Haake and Klaus (2009a,b). The stable matching correspondence is not Nash implementable in one-to-one matching markets with fewer than three agents (see Kara and Sönmez (1996)), hence our implementability result is as sharp as possible.

[5] Unlike in many-to-one matching, the set of core many-to-many matchings does not generally correspond to the set of stable many-to-many matchings (Blair (1988); see also Echenique and Oviedo (2006), Konishi and Ünver (2006)). This problem is still extant in the more general setting of many-to-many matching with contracts, hence we follow Echenique and Oviedo (2006) and Klaus and Walzl (2009) in studying a solution concept alternative to and stronger than stability. Our strengthened stability concept, *strong stability*, is stronger than the similar notion of *setwise stability* studied by Echenique and Oviedo (2006) and Klaus and Walzl (2009).

1.2 Substitutable Completability

As an application of our results for many-to-many matching with contracts, we identify a natural new condition, *substitutable completability*, sufficient for the existence of stable allocations in many-to-one contract matching markets.[6,7] If preferences are substitutably completable, then they guarantee a lattice of (many-to-one) stable allocations and, along with a form of the law of aggregate demand, are sufficient for the rural hospitals result and for the matching mechanism to be (group) strategy-proof for doctors.

1.3 Characterization of Substitutability

We show that the Hatfield and Milgrom (2005) notion of substitutability has a natural interpetation in terms of utility theory: preferences over contracts are substitutable if and only if they can be represented by a submodular indirect utility function over sets of offered contracts. This interpretation of substitutability allows us to further show an equivalence between the notion of substitutability used in auction theory and that of matching theory.

1.4 Contract Design

Previous work on bilateral matching with contracts has assumed that in every possible outcome, the relationship between each pair of agents is specified by a single contract (Crawford and Knoer (1981); Kelso and Crawford (1982); Hatfield and Milgrom (2005); Klaus and Walzl (2009)). This assumption, which Kominers (forthcoming) calls *unitarity*, implicitly restricts the set of preferences which can be considered

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[6] As we show, substitutable completability is neither weaker nor stronger than the bilateral substitutes condition of Hatfield and Kojima (2010); that is, there are substitutably completable preference profiles which do not satisfy bilateral substitutes, and there are preference profiles which satisfy bilateral substitutes but are not substitutably completable.

[7] Extension of these ideas lead to a second condition sufficient for the existence of stable allocations, which is strictly weaker than bilateral substitutes, the weakest such condition previously known.

substitutable.[8] Moreover, for many economic environments, the validity of this modeling assumption is far from apparent—even in labor markets, agents might contract separately over multiple shifts at the same employer.

For example, consider a setting with a doctor d and a hospital h. Contracts can specify one or two of the following terms: the doctor works in the morning (m); the doctor works in the afternoon (a). The doctor would most prefer to work in both the morning and the afternoon, but would be willing to work just the afternoon shift; he is unwilling to work only the morning shift. The hospital would hire the doctor for any and both shifts, but would most prefer that the doctor work only in the morning, and would rather hire the doctor full-time than for just the afternoon. We denote by x^{Π} the contract with terms $\Pi \subseteq \{m, a\}$. When morning and afternoon shifts are contracted separately, the doctor's preferences over contracts are given by

$$P_d : \{x^{\{m\}}, x^{\{a\}}\} \succ \left\{x^{\{a\}}\right\} \succ \varnothing \succ \{x^{\{m\}}\},$$

while the hospital's are given by

$$P_h : \{x^{\{m\}}\} \succ \{x^{\{m\}}, x^{\{a\}}\} \succ \left\{x^{\{a\}}\right\} \succ \varnothing.$$

There is no stable contract allocation: for the set $\left\{x^{\{m\}}, x^{\{a\}}\right\}$, the hospital will not be willing to sign $x^{\{a\}}$; for the set $\left\{x^{\{a\}}\right\}$, both parties prefer that the doctor work full time; the set $\{x^{\{m\}}\}$ is not individually rational for the doctor; and finally both parties agree that $\left\{x^{\{a\}}\right\}$ is better than no relationship at all. This lack of agreement derives from the fact that the preferences of doctor d are not substitutable–there are two contracts ($x^{\{m\}}$ and $x^{\{a\}}$) which exhibit "complementarity" for d, in the sense that d wants one ($x^{\{m\}}$) only if he has the other ($x^{\{a\}}$).

By contrast, if the parties are to negotiate over a single contract $x^{\{m,a\}}$ which

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[8] Additionally, as Echenique (forthcoming) showed, the unitarity assumption induces an embedding of the matching with contracts model into the seemingly simpler matching with salaries framework of Kelso and Crawford (1982).

encodes both the morning and afternoon shifts, agents' preferences take the forms

$$P_d : \{x^{\{m,a\}}\} \succ \varnothing, \quad P_h : \{x^{\{m,a\}}\} \succ \varnothing.$$

These preferences are substituable, and there exists a unique stable outcome $\{x^{\{m,a\}}\}$.

In this paper, we develop a theory of contract language which formalizes the intuitions from this example regarding language, substitutability, and the stability of contract allocations. This theory of language accomodates not only the setting described above, but also other natural examples such as settings with fixed costs of production (e.g., manufacturing and electricity markets). We show that market designers, when constructing the contractual language for a matching market, face a trade-off between expressiveness (i.e. the number of different contractual relationships the language can describe) and stability: the more expressive the language, the less likely it is that preferences are substitutable, and the less likely it is that a stable allocation exists.

1.5 Outline of the Paper

The remainder of this paper is organized as follows. In Sections 2 and 3, we present our basic model and review the standard terminology and solution concepts of matching with contracts. In Section 4, we discuss restrictions on preferences. There, we prove our characterizations of substitutability. We formalize our theory of contract language in Section 5, where we also discuss the relationship between language, stability, and substitutability. In Section 6, we study many-to-many matching with contracts, proving the sufficiency and necessity of substitutable preferences for the existence of stable contract allocations, along with rural hospitals and Nash implementability results; in Section 7, we apply these results to the theory of many-to-one matching with contracts. Finally, we conclude in Section 8.

Our exposition in Section 6 is essentially self-contained, so that a reader uninterested in the technical details of contract language may choose to skip Section 5.

2 Model

There are finite sets D and H of doctors and hospitals; we denote the set of all agents by $F \equiv D \cup H$. There is a set X of contracts which specify relationships between doctor-hospital pairs. We elaborate upon the structure of the set X in Section 5, but for concreteness one may think of the special case in which X takes the form $X = D \times H \times T$, for some finite set T of contractual terms. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. For a set of contracts $Y \subseteq X$, we denote $Y_D \equiv \bigcup_{y \in Y}\{y_D\}$ and $Y_H \equiv \bigcup_{y \in Y}\{y_H\}$. We denote by $x_F \equiv \{x_D, x_H\}$ the set of agents associated with contract x, and let $Y|_f \equiv \{y \in Y | f \in y_F\}$ be the set of contracts in Y associated with agent $f \in F$.

Each $f \in F$ has a strict preference relation P_f^X over sets of contracts in X involving f. For now, we take the preferences P_f^X of the agent f as given, and when the contract language X is clear from context, we will abuse notation by suppressing the superscript and writing P_f for the preference relation of f over contracts in X. In Section 5, we elaborate upon the preference relation structure, deriving P_f^X from a preference relation over contractual primitives.

For any offer set $Y \subseteq X$, we let

$$C_f(Y) \equiv \max\nolimits_{P_f} \{Z \subseteq Y | x \in Z \Rightarrow f \in x_F\}$$

be the set of contracts chosen from Y by $f \in F$.[9,10] We let

$$R_f(Y) \equiv Y - C_f(Y)$$

denote the set of contracts rejected from Y by f.

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[9] We use the term “offer set” instead of “budget set” or “set of alternatives” as agents are typically allowed to choose only one option (or point, or bundle) from a budget set. (See the defintion given on the first page of Chapter 1 of Mas-Colell et al. (1995), for instance.) Here, agents may choose any subset of contracts offered. Formally, then, we could consider the set of alternatives available to the agent as the power set of the offer set of the agent.

[10] Here, we use the notation $\max_{P_f}$ to indicate that the maximization is taken with respect to the preferences of agent f.

Let $C_D(Y) \equiv \bigcup_{d \in D} C_d(Y)$ be the set of contracts chosen from Y by doctors. The remaining contracts, rejected by all the doctors, comprise the rejected set $R_D(Y) \equiv Y - C_D(Y)$. Similarly, let $C_H(Y) \equiv \bigcup_{h \in H} C_h(Y)$ be the set of contracts chosen from Y by hospitals, and let $R_H(Y) \equiv Y - C_H(Y)$.[11]

An **allocation** is a set of contracts $Y \subseteq X$. Preference relations are naturally extended to allocations: for two allocations $Y, Z \subseteq X$, we write $Y \succ_f Z$ to mean $Y|_f \succ_f Z|_f$.

3 Solution Concepts

3.1 Stability

A key question in matching theory is whether an allocation A is stable, that is, whether there exists a blocking set of contracts Z such that all agents in $Z_F \equiv Z_D \cup Z_H$ will choose their contracts in Z from $Z \cup A$ (and possibly drop contracts in A). We codify the notion of stability in our model with the following definition:

Definition 1. An allocation A is **stable** if it is

1. **Individually rational**: for all $f \in F$, $C_f(A) = A|_f$.
2. **Unblocked**: There does not exist a nonempty **blocking set** $Z \not\subseteq A$ such that for all $f \in Z_F$, $Z|_f \subseteq C_f(A \cup Z)$.

This notion is the natural generalization of the notions of stability in the one-to-one and many-to-one matching literatures.[12] In the one-to-one matching literature, the standard definition of a stable set A requires that A be individually rational and that there be no blocking set Z such that $|Z_D| = |Z_H| = 1$; we call this **pairwise stability**. Similarly, in the many-to-one matching literature, the standard definition

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[11] Note that $R_D(Y) = \bigcup_{d \in D} R_d(Y|_d) \neq \bigcup_{d \in D} R_d(Y)$ and similarly $R_H(Y) = \bigcup_{h \in H} R_h(Y|_h) \neq \bigcup_{h \in H} R_h(Y)$.

[12] In particular, this definition is equivalent to that of Hatfield and Milgrom (2005) in the context of many-to-one matching with contracts.

of a stable set A requires that A be individually rational and that there be no blocking set Z such that $|Z_H| = 1$, where hospitals were the side of the market allowed to sign multiple contracts. If A is individually rational and there is no blocking set Z such that $|Z_H| = 1$ or $|Z_D| = 1$, we say that A is **many-to-one stable**. It is immediate that any stable allocation is many-to-one stable and that any many-to-one stable allocation is pairwise stable. Conversely, in models of one-to-one matching, any pairwise stable allocation is many-to-one stable and stable, and in models of many-to-one matching any many-to-one stable allocation is stable.

There is also a more restrictive notion of stability for many-to-many matching problems, strong stability:

Definition 2. An allocation A is **strongly stable** if it is

1. Individually rational, and
2. **Strongly unblocked**: There does not exist a nonempty set $Z \subseteq X - A$, such that for all $f \in Z_F$, there exists an individually rational Y^f such that $Z|_f \subseteq Y^f \subseteq Z \cup A$ and $Y^f \succ_f A$.

The key difference between stability and strong stability is that strong stability does not require deviations to be self-enforcing, only individually rational. It is stronger than the previously-studied *setwise stability* (Echenique and Oviedo (2006); Klaus and Walzl (2009)), a similar notion which imposes the additional requirement that the deviating agents agree on which contracts in the original allocation A to drop, i.e. for all $y \in A$, $y \in Y^d$ if and only if $y \in Y^h$ (where $y_F = \{d, h\}$).

For many-to-one (and one-to-one) matching, a matching A is stable if and only if it is strongly stable, and both of these conditions are equivalent to A being in the core. However, this is no longer true in the many-to-many matching context. Blair (1988) provides an example of a match that is stable but not strongly stable according to our definitions.[13]

3.2 Strategy-proofness

A **matching mechanism** ψ is a mapping from the set of preference profiles to the set of allocations. We examine whether certain matching mechanisms are strategy-proof; that is, whether or not it is a weakly dominant strategy for agents to truthfully reveal their preferences.

Definition 3. A matching mechanism ψ is **strategy-proof** for $s \in F$ if, for any preference profile P, $\psi(P) \succsim_s \psi(P'_s, P_{-s})$ for all P'_s. If ψ is strategy-proof for all $s \in F$, then we say that ψ is **strategy-proof**.

Similarly, we can consider the incentives of groups of individuals under a given matching mechanism.

Definition 4. A matching mechanism ψ is **group strategy-proof** for $S \subseteq F$ if, for any preference profile P, for at least one $s \in S$, $\psi(P) \succsim_s \psi(P'_S, P_{-S})$ for all P'_S. If ψ is strategy-proof for all $S \subseteq F$, then we say that ψ is **group strategy-proof**.

As is standard in the two-sided matching literature, for a matching mechanism not to be group strategy-proof, the deviation from truth-telling must make all agents in the coalition strictly better off.[14]

4 Conditions on Preferences

Because the framework of many-to-many matching with contracts places essentially no technical conditions on the content of contracts between agents, strong substantive conditions on preferences are required to guarantee the existence of stable allocations.

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[13] Note that the example of Blair (1988) is in a many-to-many matching (without contracts) context, and hence allows only one possible relationship between each pair of agents; therefore, the distinction between stability and strong stability does not hinge on the availability of multiple contracts between a pair of agents.

[14] Hatfield and Kojima (2009) discuss the motivation behind this definition.

In this section, we introduce three conditions on preferences which are central to our analysis in subsequent sections: substitutability, strong substitutability, and the law of aggregate demand.

4.1 Substitutability

We introduce three notions of substitutability, and show that they are equivalent.[15]

4.1.1 A Utility-Theoretic Definition of Substitability

We say that the indirect utility function U over offer sets **represents** preference relation P_f if

$$U(Y) > U(Z) \Leftrightarrow C_f(Y) \succ_f C_f(Z) \text{ for all } Y, Z \subseteq X.$$

That is, one offer set Y provides more utility than another offer set Z if the choice set from Y is preferred to the choice set from Z. In this context, an agent's preferences over contracts are substitutable if an additional offer is more valuable when the agent's original offer set is small. For functions over sets, this notion is captured mathematically by submodularity, and leads to the following utility-theoretic notion of substitutability:

Definition 5. Preferences of $f \in F$ are **substitutable** if they can represented by a submodular indirect utility function U over offer sets.

4.1.2 Alternative Interpretations of Substitutability

We now review two other notions of substitutability, respectively drawn from matching theory and auction theory. We show that these notions both correspond to the utility-theoretic notion of substitutability introduced above. This further unifies the matching and auction literatures, as it shows that the matching theory and auction

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[15] An analogous equivalence is obtained by Hatfield et al. (2010). The result of Hatfield et al. (2010) is distinct from ours, however, as it only applies in settings with continuously transferable utility.

theory substitutability notions are themselves equivalent. It is well-known that substitutability is a key condition for the existence of Walrasian equilibrium in auction settings with discrete goods; our equivalence results illustrate that the same structure is driving the importance of substitutability for the existence of stable matching outcomes.

In matching theory, the primary condition on preferences is substitutability, defined directly from the choice function. Intuitively, contracts x and y are substitutes if they are not complements; that is, if f rejects the contract y from Y, f will not choose y from the larger set $\{x\} \cup Y$. More formally:

Definition 6. Preferences are **matching-theory substitutable** for $f \in F$ if for all $X'' \subseteq X' \subseteq X$, $R_f(X'') \subseteq R_f(X')$.

Note that this condition applies to offer sets—it states that any contract that is rejected from an offer set X'' is also rejected from a larger offer set $X' \supseteq X''$. In fact, Definitions 5 and 6 are equivalent—the indirect utility representation exactly characterizes when preferences which are matching-theory substitutable.

Theorem 1. *The preferences of $f \in F$ are matching-theory substitutable if and only if they are substitutable.*

Intuitively, if preferences of f are not matching-theory substitutable, then there exist complementary contracts x and z, such that z is not chosen from $Y \cup \{z\}$ by f but is chosen from $\{x\} \cup Y \cup \{z\}$. But such complementary contracts would violate the definition of submodularity, as then adding z to the offer set Y would not result in a higher utility for f, but adding it to the larger offer set $\{x\} \cup Y$ would result in higher utility for f.

Via Theorem 1, we can now understand the correspondence between the notion of substitutability used in the matching literature and that used in the auction literature. In the auction literature, substitutable preferences are defined over *objects*, rather than contracts. Consider a set of unique items J. A **valuation function** $v(S)$ represents the utility (or profits) gained from having the objects in $S \subseteq J$. The **demand correspondence** for a price vector $p \in \mathbb{R}^J$ is given by

$$D(p) \equiv \arg\max_{S \subseteq J} \left\{ v(S) - \sum_{s \in S} p_s \right\}.$$

With these definitions, we can define substitutes as it is used in auction theory (see Milgrom (2004)):

Definition 7. A valuation function $v(\cdot)$ over sets of items is **auction-theory substitutable** if for all prices p, p' such that $D(p)$ is a singleton, if $j \in D(p)$ and $p' \geq p$ but $p_j = p'_j$, then $j \in D(p')$.

Intuitively, items are auction-theory substitutes if when the price of one item rises, the demand for the other items does not fall. From any price vector p, we can construct the corresponding offer set

$$O(p) \equiv \{(j, \hat{p}_j) \mid j \in J \text{ and } \hat{p}_j \geq p_j\}.$$

Given the item valuation function v of agent f, we can construct the choice function of f directly. For any p such that $D(p)$ is a singleton, let

$$\begin{aligned} C_v(O(p)) &\equiv \{(j, p_j) \mid j \in D(p)\}, \\ C_f(Y) &\equiv C_v(O(\tilde{p})) \text{ where } \tilde{p}_j = \min\{p_j \mid (j, p_j) \in Y\}. \end{aligned}$$

It is now nearly immediate that a valuation function is substitutable if and only if the preferences generated by the associated choice function over offer sets are substitutable, as it is well-known that a valuation function is substitutable if and only if the associated indirect utility function is submodular.[16]

Theorem 2. *A valuation function v is auction-theory substitutable if and only if the associated preferences over $O(p)$ are substitutable.*

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[16] For a general statement of this result, see Milgrom (2009).

The above theorem, in addition to showing the connection between the three notions of substitutability, also motivates our description of the U function as an indirect utility function: this function U corresponds to the indirect utility function of auction theory.

Given the equivalence of the three substitutability notions, we shall henceforth simply refer to all of them by the term "substitutability," which captures the key condition that the indirect utility function over contracts is submodular. The corresponding intuition is that when an agent with substitutable preferences receives new contracts, these new contracts are more valuable to him if he had fewer options before.

4.1.3 Allowing Multiple Contracts Between a Doctor-Hospital Pair

It is clear that any collection of contractual relationships between a doctor and a hospital may be bound together into a single contract. This contract structure is in fact required by Klaus and Walzl (2009), who allow at most one contract between each doctor-hospital pair.[17] However, imposing this requirement may obscure substitutable structure within agents' preferences, as the following example shows.

Consider a hospital h with two tasks, α and β. We suppose that

- doctor d can do either or both of tasks α and β;
- doctor d' can only do task β.

The hospital would like to assign a doctor to task α, but would prefer that d', rather than d, complete task β. These are very natural preferences,[18] and intuitively they

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[17] Within the theory of contract language we develop in Section 5, this restriction is imposed by taking the contract language X to be such that if $x \in X$ and $y \in X$, then $(x \cup y) \in X$, and adding the additional requirement that preferences over contracts cover only the maximally-bundled expressions of each collection of primitives. The latter requirement is mathematically innocuous, and we will maintain it for expositional clarity.

[18] Perhaps β is a specialist task, such as radiology work, while α is a general-practice task. Doctor d is a general practitioner, and hence can perform both tasks α and β. Meanwhile, doctor d' is a specialist, and therefore can only perform task β, but can perform β better than d can. In the

should be substitutable, as d' "substitutes" for d in performing task β. But if we allow at most one contract per hospital-doctor pair, then the possible assignments of doctor d take the form of three possible contracts between d and h:

- $x^{\{(d,\alpha)\}}$, where d performs only task α,
- $x^{\{(d,\beta)\}}$, where d performs only task β, and
- $x^{\{(d,\alpha),(d,\beta)\}}$, where d performs both tasks.

Using similar notation, we let $x^{\{(d',\beta)\}}$ be the contract between d' and h that specifies that d' performs task β. With these contracts available, the preferences of h would be

$$P_h : \left\{x^{\{(d,\alpha)\}}, x^{\{(d',\beta)\}}\right\} \succ \left\{x^{\{(d,\alpha),(d,\beta)\}}\right\} \succ \left\{x^{\{(d,\alpha)\}}\right\} \succ \left\{x^{\{(d',\beta)\}}\right\} \succ \left\{x^{\{(d,\beta)\}}\right\} \succ \varnothing$$

which are not substitutable, as

$$R_h\left(\left\{x^{\{(d,\alpha),(d,\beta)\}}, x^{\{(d,\alpha)\}}\right\}\right) = \left\{x^{\{(d,\alpha)\}}\right\} \not\subseteq \left\{x^{\{(d,\alpha),(d,\beta)\}}\right\} = R_h\left(\left\{x^{\{(d,\alpha),(d,\beta)\}}, x^{\{(d,\alpha)\}}, x^{\{(d',\beta)\}}\right\}\right).$$

However, if we allow multiple contracts between agent pairs[19] the preferences of h can be written in the substitutable form

$$P'_h : \left\{x^{\{(d,\alpha)\}}, x^{\{(d',\beta)\}}\right\} \succ \left\{x^{\{(d,\alpha)\}}, x^{\{(d,\beta)\}}\right\} \succ \left\{x^{\{(d,\alpha)\}}\right\} \succ \left\{x^{\{(d',\beta)\}}\right\} \succ \left\{x^{\{(d,\beta)\}}\right\} \succ \varnothing.$$

This rewritten preference relation makes clear the intuitive fact that d' substitutes for d in performing task β. Without the presence of multiple contracts between the doctor-hospital pair $(d,h) \in D \times H$, this intuition is obscured, as is the fact (implied

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terminology for contract language which we introduce in Section 5, (d,α), (d,β) and (d',α) are contractual primitives, and

$$\pi(d,h) = \{(d,\alpha),(d,\beta)\}, \quad \pi(d',h) = \{(d',\beta)\}.$$

[19] In the terminology of Section 5, this is equivalent to working in a coarser language in which the assignment of α and β to d is expressed as $\{x^{\{(d,\alpha)\}}, x^{\{(d,\beta)\}}\} = \{\{(d,\alpha)\},\{(d,\beta)\}\}$.

by our existence result, Theorem 10) that stable allocations exist under P_h so long as the preferences of d and d' are substitutable.

In our subsequent discussion, we assume the possibility of multiple contracts between doctor-hospital pairs.[20] As the example just presented suggests, the class of substitutable preferences in our framework therefore includes many sets of preferences which are naturally substitutable but were not considered substitutable in previous many-to-many matching models.

4.2 Strong Substitutability

In addition to substitutability, we will discuss the implications of a more restrictive condition on preferences, called strong substitutability. Echenique and Oviedo (2006) introduced this condition as a strengthening of substitutability, and Klaus and Walzl (2009) then extended it to the setting of many-to-many matching with contracts.

Definition 8. Preferences are **strongly substitutable** for $f \in F$ if for all $X'', X' \subseteq X$ such that $C_f(X'') \succ_f C_f(X')$, $X' \cap C_f(X'') \subseteq C_f(X')$.

Intuitively, strong substitutability means that if an agent f chooses contract y from a set of contracts Y, and if $y \in Y'$ and Y' is a "worse" set of contracts available for f than Y is, then f still chooses y from Y'.

4.3 The Law of Aggregate Demand

Finally, we introduce an additional condition on preferences: the law of aggregate demand, first introduced by Hatfield and Milgrom (2005).[21]

Definition 9. The preferences of $f \in F$ satisfy the **law of aggregate demand** if for all $X'' \subseteq X' \subseteq X$, $|C_f(X'')| \leq |C_f(X')|$.

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[20] This is a substantive assumption on the contract set X, but a very weak one.

[21] Alkan and Gale (2003) introduced a related condition called "size monotonocity."

The law of aggregate demand states that if the choice set of an agent f expands, that agent chooses (weakly) more contracts.[22]

5 Contract Language

We now develop a theory of the contract set X as a language for expressing bundles of underlying primitive contract terms. Throughout this section, we allow the contract set to vary, and discuss the effects of varying contract language on stability and substitutability of preferences.

5.1 Basic Theory of Language

For each doctor-hospital pair $(d, h) \in D \times H$, there is a set of **contractual primitives** $\pi(d, h)$ which defines the set of possible contractual relationships between d and h. We write

$$\Pi_d \equiv \bigcup_{h \in H} \pi(d, h)$$

for the set of primitives associated to doctor $d \in D$ and

$$\Pi_h \equiv \bigcup_{d \in D} \pi(d, h)$$

for the set of primitives associated to hospital $h \in H$. We require that $\pi(d, h) \cap \pi(d', h') = \varnothing$ for all $(d, h) \neq (d', h')$ so that each primitive uniquely identifies a doctor and hospital.

A **primitive allocation** is a collection of primitives $\Lambda \subseteq \bigcup_{(d,h) \in D \times H} \pi(d, h)$.

A **contract** between d and h is a collection of primitives in $\pi(d, h)$. Denoting the power set of $\pi(d, h)$ by $\mathcal{P}(\pi(d, h))$, a contract between d and h is just a nonempty element of $\mathcal{P}(\pi(d, h))$. For example, $\pi(d, h)$ might consist of all the distinct work

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[22] We note here that one can show a slightly stronger result characterizing the law of aggregate demand than in Hatfield and Milgrom (2005). In particular, let $X = D \times H \times T \times W$, where T encodes the non-wage terms of the contract and W specifies wages. Then, if the preferences of hospital h are quasilinear and are substitutable over $Y \equiv D \times H \times T$, then those preferences satisfy the law of aggregate demand. This result follows using the proof approach of Hatfield and Milgrom (2005).

hours available at hospital h in a given week; a contract between h and d is a subset of $\pi(d,h)$ corresponding to the work hours assigned to d by h.

A **contract language** $X_{(d,h)}$ **for** $(d,h) \in D \times H$ is a set of contracts between d and h, i.e. a subset of $\mathcal{P}(\pi(d,h)) - \{\emptyset\}$. More generally, a **contract language** X is a union of contract languages for each agent pair: $X = \bigcup_{(d,h)\in D\times H} X_{(d,h)}$ with $X_{(d,h)} \subseteq \mathcal{P}(\pi(d,h)) - \{\emptyset\}$ for each $(d,h) \in D \times H$. We say that a primitive allocation Λ is **expressible** in the contract language X if there exists some $Y \subseteq X$ such that $\Lambda = \bigcup_{y\in Y} y$. In this case we say that Y **expresses** Λ.

Each $f \in F$ has a strict preference relation P_f over the set $\mathcal{P}(\Pi_f)$ of bundles of primitives involving f. For any contract language X, this preference relation over primitives induces a preference relation, denoted P_f^X, over bundles of contracts in X (subsets of $\mathcal{P}(X)$). This induced preference relation is not strict, but its only indifferences arise on bundles of contracts $Y, Y' \subseteq X$ which correspond to the same primitive allocation ($\bigcup_{y\in Y} y = \bigcup_{y'\in Y'} y'$). When describing preferences in the sequel, despite indifference between these **primitive-equivalent** sets of contracts, we will typically assume that the preference relation P_f^X is strict, arbitrarily breaking ties among primitive-equivalent contract sets.[23] The choice correspondence associated to P_f^X is denoted by C_f^X. As before, when the contract language X is clear from context, we will abuse notation by suppressing the superscript and writing P_f for the preference relation of f over contracts in X, and C_f for the associated choice correspondence.

When $\pi(d,h)$ is a singleton for each doctor-hospital pair $(d,h) \in D \times H$, and $X = \bigcup_{f\in F} \Pi_f$, we recover the many-to-many matching model considered by Sotomayor (1999), Konishi and Ünver (2006), and Echenique and Oviedo (2006). In this case, each primitive allocation is exactly a (many-to-many) matching between doctors and

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[23] This choice is not entirely without loss of generality—it affects the set of stable allocations. However, arbitrary tie-breaking is not problematic, as if for a given tie-breaking of indifferences over primitive-equivalent expressions of a primitive allocation Λ, the allocation $Y \subseteq X$ stably expresses Λ, then for any tie-breaking there is a (possibly distinct) allocation $Y' \subseteq X$ which stably expresses Λ.

hospitals.

Although our model can recapture the familiar structure of many-to-many matching, its more general structure exhibits a key distinction from classical matching models: depending upon the structure of the contract language X, some primitive allocations are not expressible at all, and others may *only* be expressible if doctors $d \in D$ and hospitals $h \in H$ are allowed to sign multiple contracts with each other to describe their mutual obligations. This latter feature stands in sharp contrast to the restriction adopted by Klaus and Walzl (2009) that each doctor-hospital pair sign at most a single contract. As we saw in Section 4.1.3 and illustrate in more generality below, the ability of doctors to sign multiple contracts with the same hospital has subtle implications for the definition of substitutes.

5.2 Language and Stability

If a primitive allocation Λ is expressible in the contract language X by a stable allocation Y, then we say that Λ is **stable with respect to the contract language** X.[24]

It is clear that primitive allocations may be stable with respect to some contract languages and unstable with respect to others: for example, the empty allocation is stable with respect to an empty contract language, but is generally unstable once contracts with content are allowed. We now formalize and extend the structure behind this observation.

To facilitate comparisons between languages, we introduce the following partial order on contract languages:

Definition 10. A contract language X **is finer than** (or **refines**) another contract language X' if $X \supsetneq X'$. In this case, we also say that X' **is coarser than** (or **coarsens**) X and write $X \rhd X'$.[25]

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[24] Unfortunately, although agents are indifferent over contract sets which express the same primitive allocations, not all expressions of a primitive allocation Λ stable with respect to X need be stable.

[25] Of course, any (strict) subset of a contract language X coarsens X. Although we could simply

Refinement of a language X' corresponds to an increase in **expressiveness**: if $X \rhd X'$, then each agent may express a richer preference relation over contracts in X than she can over contracts in X'.[26]

With this ordering $\rhd$, the set of contract languages forms a lattice, with least upper bound and greatest lower bound respectively given by the setwise union and intersection operations. We quickly observe a tradeoff between the expressiveness of a language and the stability of underlying allocations: finer languages allow more complex preference specification, which leads to (weakly) reduced stability.

Theorem 3. *Suppose that $X \rhd X'$ and that an allocation $Y \subseteq X'$ is (strongly) stable in X. Then, Y is (strongly) stable in X'.*

Theorem 3 shows the natural result that coarsening a contract language X preserves the stability of an allocation Y, so long as Y is not eliminated from the language. However, this result applies only to allocations, not to primitive allocations. To see this, consider a setting with a single doctor and hospital and two contractual primitives: the doctor working (w) and being compensated (\$). Formally, we write $D = \{d\}$, $H = \{h\}$, and $\pi(d,h) = \{w, \$\}$. We suppose that agents' underlying preferences take the natural form

$$P_d : \{\{\$\}\} \succ \{\{w, \$\}\} \succ \varnothing, \quad P_h : \{\{w\}\} \succ \{\{w, \$\}\} \succ \varnothing.$$

Both agents want to contract, but the doctor would most prefer to be paid for nothing, and the hospital would most prefer that the doctor work for free. As before, we denote $x^{\Pi} \equiv \Pi$ for a set of primitives Π. When all contracts are possible—$X =$

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denote the refinement relation by the (strict) setwise inclusion relation $\supsetneq$, we use the distinguished notation $\rhd$ to help clarify when we are actively comparing two contract languages.

[26] We need not have $P_f^X \neq P_f^{X'}$ for all $f \in F$, since X might only differ from X' by the addition of contracts disjoint from Π_f.

$\{x^{\{w\}}, x^{\{\$\}}, x^{\{w,\$\}}\}$—preferences over contracts are

$$P_d^X : \{x^{\{\$\}}\} \succ \{x^{\{w,\$\}}\} \sim \{x^{\{w\}}, x^{\{\$\}}\} \succ \varnothing,$$
$$P_h^X : \{x^{\{w\}}\} \succ \{x^{\{w,\$\}}\} \sim \{x^{\{w\}}, x^{\{\$\}}\} \succ \varnothing,$$

and the unique stable allocation is $\{x^{\{w,\$\}}\}$. If we coarsen X to $X' = \{\varnothing, x^{\{w\}}, x^{\{\$\}}\}$ by removing the contract $x^{\{w,\$\}}$, the preferences reduce to

$$P_d^{X'} : \{x^{\{\$\}}\} \succ \{x^{\{w\}}, x^{\{\$\}}\} \succ \varnothing,$$
$$P_h^{X'} : \{x^{\{w\}}\} \succ \{x^{\{w\}}, x^{\{\$\}}\} \succ \varnothing,$$

under which only $\varnothing$ is stable.[27] Thus, we see that the stability of the primitive allocation $\{\{w, \$\}\}$ is not preserved under coarsening.

A natural assumption when multiple contracts are allowed between a doctor and hospital is that if a doctor chooses to abrogate one of his contracts with a particular hospital, he must abrogate all contracts with that hospital. Hence, a natural question is whether, if an allocation Λ is stable with respect to a contract language X, then is it still stable if, when we consider a deviation such that a doctor drops some contract with a hospital, we impose that the doctor must drop all contracts with that hospital (and vice versa). Our next result demonstrates that this particular *noncoarsening* modification of a contract language X preserves the stability of underlying primitive allocations: the stability of a primitive allocation Λ is preserved when the (d, h)-primitives in Λ are bound together into a single contract. This result is essential for the implementation of many-to-many contract matching markets, as it demonstrates that, in any stable allocation, the full contractual relationship between a doctor and hospital may be combined into a single contract without reducing stability.

Theorem 4. *Suppose that the primitive allocation Λ is stable with respect to the contract language X, and let $Y \subseteq X$ be a set of contracts in X which stably expresses Λ.*

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[27] Note that $\varnothing$ is not blocked by $\{x^{\{w\}}, x^{\{\$\}}\}$ as $C_d^{X'}(\{x^{\{w\}}, x^{\{\$\}}\}) = \{x^{\{w\}}\} \neq \{x^{\{w\}}, x^{\{\$\}}\}$.

Now, let

$$X' \equiv (X - Y) \bigcup \left(\cup_{(d,h) \in D \times H} \{ \Pi_d \cap \Lambda \cap \Pi_h \} \right)$$

be the contract language such that each contracting relationship in Y *is expressed by a singleton contract. Then,* Λ *is stable with respect to* X'*.*

5.3 Language and Substitutability

Substitutability is a stringent condition, in practice, and certainly need not be true for agents' underlying preferences over primitives. However, we observe that substitutability is more likely to arise in coarser languages, and so clever contract language design can lead to substitutable preferences.

For example, consider a setting with a single doctor, a single hospital, and two contractual primitives: working the morning shift (m) and working the afternoon shift (a). Formally, we write $D = \{d\}$, $H = \{h\}$, and $\pi(d, h) = \{m, a\}$. Suppose that the agents' underlying preferences over primitives are

$$P_d : \{\{m, a\}\} \succ \varnothing, \quad P_h : \{\{m, a\}\} \succ \varnothing.$$

Both agents want to contract over a full-time job, but neither will contract over a part-time position. If m and a are split into separate "part-time job" contracts $x^{\{m\}}$ and $x^{\{a\}}$, then the agents' preferences are not substitutable—$x^{\{m\}}$ and $x^{\{a\}}$ are complements in this language. This is true even if a single "full-time job" contract $x^{\{m,a\}}$ is available in addition to the part-time contracts. By contrast, if only the full-time contract $x^{\{m,a\}}$ is available, agents' preferences are substitutably expressed as

$$P_d^{\{x^{\{m,a\}}\}} : \{x^{\{m,a\}}\} \succ \varnothing, \quad P_h^{\{x^{\{m,a\}}\}} : \{x^{\{m,a\}}\} \succ \varnothing.$$

Every contract language X has a coarsening X' over which preferences are substitutable. Our next result shows that once such a coarsening X' is found, any further coarsening of X' will induce substitutable preferences as well.

Theorem 5. *Suppose that $X \rhd X'$ and that the preference relation P_f^X of an agent $f \in F$ is (strongly) substitutable. Then, $P_f^{X'}$ is (strongly) substitutable, as well.*

Just as Theorem 3 indicates a tradeoff between expressiveness and stability, Theorem 5 indicates a tradeoff between expressiveness and substitutability. Our later results (Theorems 10 and 12) show that substitutability of preferences is necessary and sufficient for the existence of stable allocations, hence Theorem 5 implies a direct tradeoff between expressiveness and the existence of stable allocations. However, the "optimal" language seems potentially difficult to compute in practice, as it depends upon parameters which the market designer must assess.

6 Many-to-Many Matching with Contracts

Throughout this section, we return to a fixed contract language X and the standard notation and terminology of matching with contracts.[28] We generalize key results of matching theory to the setting of many-to-many matching with contract language X, and discuss the how the significance of these results depends upon the structure of X. Then, as an application of our results for many-to-many matching with contracts, we generalize a series of previous results for many-to-one matching with contracts.

6.1 Substitutes and Stability

6.1.1 Existence of Stable Allocations

To prove the existence of a stable allocation, we examine the operator Φ defined by

$$\Phi(Y) = X - R_H\left(X - R_D(Y)\right),$$

a generalized version of the deferred acceptance algorithm of Gale and Shapley (1962). If we consider the set Y to be the set of contracts available to the doctors, then $X - R_D(Y)$ is the set of contracts that is not rejected by the doctors; hence, it is

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[28] As we mentioned in Section 4.1.3, we do make the weak assumption that X allows for the possibility of multiple contracts between doctor-hospital pairs.

the set available to the hospitals. Similarly, we can consider $X - R_H\left(X - R_D(Y)\right)$ to be the set of contracts that are now available to the doctors after this round of the algorithm. Hence, at any iteration of the operator, $C_D(Y)$ is the set of contracts "held" by the doctors, and $C_H\left(X - R_D(Y)\right)$ is the set of contracts "held" by the hospitals. A fixed point exactly corresponds to the case when doctors and hospitals are "holding" the same set of contracts. We codify this intuition in the following lemma.

Lemma 6. *If* $C_h\left(X - R_D(Y)\right) \neq C_D(Y)|_h$ *for some hospital* h*, then* Y *is not a fixed point of* Φ*.*

It will also be helpful to understand the structure of blocking sets in the presence of substitutable preferences. When preferences are substitutable, we see that any contract from any blocking set comprises a blocking set in and of itself.

Lemma 7. *Suppose* Z *is a blocking set for* Y*, and that the preferences of all agents are substitutable. Then for any* $z \in Z$*,* $\{z\}$ *is a blocking set for* Y*.*

Lemma 7 shows that when preferences are substitutable, it is enough to consider blocking sets which involve only one contract. This is intuitive: when there are no complementarities between contracts, each contract in a blocking set Z must have its own blocking power—no contract can rely upon the presence of other contracts in Z. This is similar to and generalizes previous results (e.g., Lemma 5.5 of Roth and Sotomayor (1990)) showing that when preferences are substitutable, blocking sets can be "boiled down" to a single blocking contract.

We can now show that, if preferences are substitutable, then the fixed points of the map Φ correspond to stable allocations.

Theorem 8. *Suppose that* $\Phi(Y) = Y$*, and that preferences of all agents are substitutable. Then,* $C_D(Y)$ *is a stable allocation.*

The proof of this result is intuitive, and follows the deferred acceptance intuition presented earlier. The fixed points of Φ correspond to the cases when doctors and hospitals are holding the same set of contracts. With substitutable preferences, this is sufficient to imply stability. Under the action of Φ, neither a doctor nor a hospital will "hold" a contract set which is not individually rational. If a blocking set exists, then by Lemma 7 we can identify a single blocking contract $z \notin C_D(Y)$; this contract will not be rejected if "proposed" under Φ, and so Y cannot be a fixed point.

Note that, unlike in the many-to-one matching case (see Theorem 1 of Hatfield and Milgrom (2005)), the conclusion of Theorem 8 need not hold when preferences are not substitutable. For example, consider the following preferences:

$$h : \{x, \tilde{x}\} \succ \varnothing,$$

$$d : \{x, \tilde{x}\} \succ \varnothing.$$

The only stable set is $\{x, \tilde{x}\}$, but the sets $\{x\}$ and $\{\tilde{x}\}$ are both fixed points of Φ.

However, to completely characterize the set of stable allocations, we must know whether there are stable allocations which are not fixed points of Φ. It turns out that, when all agents' preferences are substitutable, there are none—every stable allocation is a fixed point of Φ.

Theorem 9. *Suppose $\hat{Y}$ is stable and all agents' preferences are substitutable. Then there exists a set of contracts $Y \subseteq X$ such that $\Phi(Y) = Y$ and $C_D(Y) = \hat{Y}$.*

The proof of Theorem 9 proceeds in two steps. First, substitutability of doctor preferences is used to show that there exists a largest set of contracts $Y \subseteq X$ such that $C_D(Y) = \hat{Y}$. Then, substitutability of hospital preferences and the stability of $\hat{Y}$ are used to prove that $C_H(X - R_D(Y)) = \hat{Y}$—doctors and hospitals want to hold exactly the same set of contracts from Y. Intuitively, Y is the largest set of contracts from which all doctors want their contracts of $\hat{Y}$, and since $\hat{Y}$ is stable, hospitals will accept exactly the doctors' offers from $C_D(Y) = \hat{Y}$.

Together, Theorems 8 and 9 show that, when preferences are substitutable, there is a bijective correspondence between the set of fixed points of Φ and the set of stable allocations. Furthermore, Φ is clearly isotone when agents' preferences are substitutable, in the sense that if $Y' \subseteq Y$, then $\Phi(Y') \subseteq \Phi(Y)$. Hence, by Tarski's theorem, there exists a lattice of fixed points of this operator, where the ordering is given by set inclusion. This allows us to prove that substitutability on both sides of the market is sufficient for the existence of stable allocations.

Theorem 10. *If all agents' preferences are substitutable, then there exists a nonempty finite lattice of fixed points of Φ; these fixed-points correspond exactly to the set of stable allocations. The largest fixed point of Φ corresponds to the doctor-optimal stable allocation, and the smallest fixed point corresponds to the doctor-pessimal stable allocation.*

We observe that the structure of stable allocations in many-to-many matching with contracts is the same as in prior models of matching. Specifically, the lattice structure of stable matches ensures that there is "opposition of interests," that is, if any stable allocation Y is preferred by all the doctors to a different stable allocation Y', then all the hospitals prefer Y' to Y. In particular, this tells us that hospital-optimal and hospital-pessimal stable allocations exist, and that they are the doctor-pessimal and doctor-optimal stable allocations, respectively.[29]

Corollary 11. *Suppose that preferences of all agents are substitutable, and consider two stable allocations Y and Y'. If $Y \succsim_d Y'$ for all $d \in D$, then $Y' \succsim_h Y$ for all $h \in H$. In particular, the doctor-pessimal stable allocation is the hospital-optimal stable allocation, and the doctor-optimal stable allocation is the hospital-pessimal stable allocation.*

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[29] Opposition of interests along these lines was first identified by Roth (1984b), and has been generalized in several settings including that of Hatfield and Milgrom (2005).

6.1.2 Necessity of Substitutable Preferences

From the preceding analysis, it is clear that substitutability is a sufficient condition for the existence of a stable allocation. However, it is well-known that in the model of many-to-one matching with contracts, weaker conditions on preferences can be found that guarantee the existence of a stable allocation.[30] Surprisingly, these results for weakened substitutes conditions do *not* carry over to the many-to-many matching with contracts model.

In particular, we show that if there are at least two agents of each type and some agent's preferences violate substitutes, then substitutable preferences for the other agents can be constructed such that no stable allocation exists.

Theorem 12. *Suppose the preferences of some agent f are not substitutable, there are at least two other agents of each type, and that X contains at least one contract between every doctor-hospital pair. Then there exist substitutable preferences for the other doctors and hospitals such that no many-to-one stable allocation exists.*[31]

If the preferences of a hospital h are not substitutable, then there exist contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ (with $z \notin Y$) such that

$$z \notin C_h(Y \cup \{z\})$$
$$z \in C_h(\{x\} \cup Y \cup \{z\}.)$$

The proof of Theorem 12 proceeds in two cases, depending on whether $x_D \neq z_D$ or $x_D = z_D$. In the first case, x_D and z_D are taken to have opposing preferences over contracts with hospitals h and h' (with x_D preferring h'). Hospital h' prefers z_D, so

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[30] See Hatfield and Kojima (2010). Note that this exception holds only in models with contracts. In particular, if there is a unique contract between each doctor-hospital pair, Hatfield and Kojima (2008) show that substitutability is the sufficient and necessary condition to guarantee the existence of a stable match.

[31] Note that this result implies that, under the assumptions of the theorem, there exist substitutable proferences for the doctors and hospitals other than f such that no stable allocation exists, as stability is a more restrictive notion than many-to-one stability.

that whenever z_D offers a contract with h', he blocks x_D from doing so. By so doing, x_D consents to work for h, who in turn now wishes to take on contract z. However, this opens up the position at h', and now x_D no longer wishes to work at h. The intuition for the second case is similar, although the technical details differ.

The viability of many-to-many allocations is crucial to this argument as the proof requires that doctors in Y_D (other than x_D and z_D) be willing to accept any and all contracts offered to them. Since in principle X can contain multiple contracts with each doctor, the doctors in Y_D must in general be willing to accept multiple contracts. This argument cannot be recovered in a many-to-one matching setting. In each of these instances, the distinction of our setting from that of Klaus and Walzl (2009)—that doctors may desire multiple contracts with the same hospital—is directly relevant.

Weakening the solution concept beyond many-to-one stabilty may assuage the difficulty presented in Theorem 12, but may be otherwise unsatisfactory. For example, considering pairwise stability is not very helpful, as many allocations are pairwise stable that, intuitively, we would not expect to be stable in practice. Furthermore, even preferences that are very complementary, such as when a hospital either desires nothing or exactly two contracts, allow for pairwise stable allocations. Consider the following preferences:

$$\begin{array}{ll} P_h : \{x, z\} \succ \varnothing & \quad P_{x_D} : \{x'\} \succ \{x\} \succ \varnothing \\ P_{h'} : \{z'\} \succ \{x'\} \succ \varnothing & \quad P_{z_D} : \{z\} \succ \{z'\} \succ \varnothing. \end{array}$$

For these preferences, $\{z'\}$ is a pairwise stable allocation, as any block involving h must include contracts with both x_D and z_D. Nevertheless, we would not expect such an allocation to be stable and, indeed, there are no (many-to-one) stable allocations for these preferences.

6.1.3 Couples Matching

There is a great deal of interest in the question of when stable matches are guaranteed to exist, even when couples are present (e.g., Kojima (2007a), Klaus et al. (2007), Klaus and Klijn (2007)), and the answers to this question are of practical importance for real world applications such as the NRMP (Roth and Peranson (1999)). Many previous studies of matching with couples have, for simplicity, assumed that the hospitals have singleton preferences while couples may desire two positions. However, for applications such as the NRMP, hospitals commonly have preferences over how to fill many positions.

Our work shows that the previous literature understates the difficulty of finding stable couples matchings, as for hospitals with more realistic, substitutable preferences, the class of substitutable prefernces is the most general class of preferences for couples of doctors under which a stable match is guaranteed to exist. Furthermore, substitutability is an extremely restrictive (and unrealistic) condition as applied to preferences of couples: it implies that each member of the couple will take the best option available to him or her, regardless of the position of the other, so it is as if assuming the couple were simply two seperate doctors.

6.1.4 The Law of Aggregate Demand

We now examine which additional results can be shown when preferences satisfy the law of aggregate demand. We first show an analogue of the rural hospitals theorem of Roth (1984a).

Theorem 13. *If preferences are substitutable and satisfy the law of aggregate demand, then each agent signs the same number of contracts at every stable allocation.*

This theorem is an immediate and elegant consequence of the law of aggregate demand and the lattice structure obtained in Corollary 11. Since for any stable allocation A, every hospital prefers A to the doctor-optimal stable allocation A^*, the law of aggregate demand guarantees that $|A^*| \leq |A|$. But no doctor can receive strictly more contracts at A^* than at A unless some other doctor receives strictly fewer contracts at A^* than at A. This cannot happen because every doctor is weakly better off at A^* than at A, and every doctor's preferences satisfy the law of aggregate demand.

Theorem 13 tells us that each agent will have the same number of contracts in every stable allocation. However, if the structure of X is irregular, the implications of this result may be unclear. Consider the example in which $D = \{d\}$, $H = \{h\}$ and contracts denote work shifts of different lengths: $X = \left\{x^{\{20\}}, x^{\{40\}}\right\}$ where $x^{\{t\}}$ encodes a t-hour work shift for doctor d at hospital h. In this case, it is unclear what is meant if the total number of contracts signed by h is invariant across stable allocations—the total number of hours worked at h might nonetheless change.[32] In the terminology of contract language, this problem occurs because the contracts represent different numbers of primitive work-units (in this case, twenty-hour shifts). When all contracts are denoted in a fixed unit, however, Theorem 13 has a natural interpretation: each agent receives the same amount of work at every stable allocation.

It is also known that, in the presence of substitutable preferences, the law of aggregate demand is a necessary condition for Theorem 13 to hold, in the sense that if any agent's preferences do not satisfy the law of aggregate demand, then substitutable preferences satisfying the law of aggregate demand can be constructed for the other agents such that the conclusion of the rural hospitals theorem fails (see Theorem 9 of Hatfield and Milgrom (2005)).

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[32] To see this, suppose that preferences are given by

$$P_h : \left\{x^{\{20\}}\right\} \succ \left\{x^{\{40\}}\right\} \succ \varnothing,$$
$$P_d : \left\{x^{\{40\}}\right\} \succ \left\{x^{\{20\}}\right\} \succ \varnothing.$$

Then both $\left\{x^{\{20\}}\right\}$ and $\left\{x^{\{40\}}\right\}$ are stable allocations which correspond to distinct numbers of total work-hours.

6.1.5 Strategy-Proofness and the Weak Pareto Property

The law of aggregate demand is also the key condition for two other additional results in the many-to-one matching literature: one-sided strategy-proofness and weak Pareto optimality (Hatfield and Milgrom (2005), Kojima (2007b), Hatfield and Kojima (2010)). One-sided strategy-proofness implies that the mechanism that chooses the doctor-optimal (hospital-optimal) stable match is strategy-proof for the doctors (hospitals). Weak Pareto optimality for the doctors (hospitals) states that there does not exist an individually rational matching that all doctors (hospitals) strictly prefer to the doctor-optimal (hospital-optimal) stable match. Unfortunately, these results do not carry over to the context of many-to-many matching. Indeed, Theorems 5.10 and 5.14 of Roth and Sotomayor (1990) provide an example where the unique stable match is not weakly Pareto optimal for the hospitals. Furthermore, in that example, one hospital has an incentive to misstate its preferences, although the mechanism chooses the hospital-optimal stable match. Finally, we note that these negative results still hold even if we require that hospital preferences both be strongly substitutable and satisfy the law of aggregate demand.[33]

6.1.6 Nash Implementability of Stable Allocations

Our next theorem shows that the stable allocation correspondence is Nash implementable whenever it is nonempty and there are at least three agents. Informally, this means that all stable allocations can be achieved non-cooperatively, through strategic interactions in equilibrium. The requirement of three agents is as sharp as possible, since Kara and Sönmez (1996) have already proven that the stable matching correspondence is not Nash implementable in the setting of one-to-one matching when there are fewer than three agents.

First, we review some standard terminology and notation. A **generalized match-**

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[33] The preferences given in the example of Roth and Sotomayor (1990) can be shown to be strongly substitutable for an appropriate choice of preferences over doubleton sets.

ing mechanism is a pair $(\mathcal{M}, o)$, where $\mathcal{M} \equiv \prod_{f \in F} \mathcal{M}_f$ denotes a set of strategy profiles and o is an **outcome function** mapping strategy profiles into allocations.[34] As is standard, we identify a mechanism $(\mathcal{M}, o)$ with its outcome function, o. For a given profile P of agents' true preferences, a mechanism o induces a non-cooperative strategic form game $\Gamma_o(P)$, in which the outcome $o(m)$ of a strategy profile $m \in \mathcal{M}$ is evaluated using agents' true preferences.

We write $\mathrm{NE}(\cdot)$ for the Nash equilibrium correspondence. A mechanism o is said to **Nash implement solution** φ if, for all possible profiles P,

$$\varphi(P) = o(\mathrm{NE}(\Gamma_o(P))).$$

That is, o Nash implements φ if the set of allocations in $\varphi(P)$ are exactly those which are the outcomes (under o) of Nash equilibria of $\Gamma_o(P)$.

Now, we state our implementability result:

Theorem 14. *If $|F| \geq 3$, then the stable allocation correspondence is Nash implementable whenever it is nonempty.*

Theorem 14 subsumes the analogous results of Kara and Sönmez (1996, 1997) and Haake and Klaus (2009a,b) for less-general matching settings. The proof of Theorem 14 is a straightforward generalization of the argument used by Haake and Klaus (2009a) in the setting of many-to-one matching with contracts, hence we omit it.[35]

Combining Theorem 14 with Theorem 10 shows in particular that the stable allocation correspondence is Nash implementable when all agents' preferences are

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[34] We use the adjective "generalized" to indicate that, unlike in the "matching mechanisms" introduced in Section 3.2, here we consider as input a generalized space of strategies rather than the space preference profiles.

[35] In fact, the argument follows that of Haake and Klaus (2009a) directly, but is slighlty simpler in our framework. Specifically, the first subargument of Step 3 in the proof given by Haake and Klaus (2009a) can be omitted, since in our framework both doctors and hospitals may accept multiple contracts.

substitutable. An additional consequence of Theorem 14 is that the stable matching correspondence is monotonic in the sense of Maskin (1999).

6.2 Strong Substitutes and Strong Stability

We now show that if preferences on one side of the market are strongly substitutable, and if those on the other side are substitutable, then any stable match is strongly stable. This result generalizes the analogous results of Echenique and Oviedo (2006) and Klaus and Walzl (2009) for less-general matching models.

Theorem 15. *If all agents' preferences are substitutable, and if furthermore the preferences of all agents of one type are strongly substitutable, then an allocation is stable if and only if it is strongly stable.*

Combining Theorem 15 with Theorems 10 and 14, we see that if all agents' preferences are substitutable, and if the preferences of all agents of one type are strongly substitutable, then

- strongly stable allocations exist, and
- the strongly stable matching correspondence is Nash implementable.

Unfortunately, in contrast to our results for substitutable preferences and stable allocations, strongly substitutable preferences are not necessary for the existence of strongly stable allocations. Indeed, if the preferences of hospital h are given by

$$P_h : \{x, y\} \succ \{x, z\} \succ \{x\} \succ \{y\} \succ \{z\},$$

then it is not possible to give substitutable preferences for the doctors and strongly substitutable preferences for hospitals other than h such that no strongly stable match exists.

However, without strongly substitutable preferences, the existence of strongly stable allocations is not guaranteed. For example, consider the preferences

$$\begin{aligned}
P_h &: \{\hat{z}, y\} \succ \{\hat{z}, z\} \succ \{\hat{z}\} \succ \{y\} \succ \{z\} \\
P_{h'} &: \{\hat{z}'\} \\
P_{z_D} &: \{\hat{z}', z\} \succ \{\hat{z}, z\} \succ \{z\} \succ \{\hat{z}'\} \succ \{\hat{z}\} \\
P_{y_D} &: \{y\}\,.
\end{aligned}$$

Here, only the preferences of h and z_D are not strongly substitutable, but the only stable match—$\{y, \hat{z}'\}$—is not strongly unblocked—to see this, take $Z = \{\hat{z}, z\}$ in the definition of strong unblockedness.

7 Completability and Many-to-One Matching with Contracts

As an application of our results for many-to-many matching markets, we now describe a method for "completing" preferences of hospitals in settings where each doctor desires at most one contract, i.e. the setting of many-to-one matching with contracts (Hatfield and Milgrom (2005)). Intuitively, a completion of the preference relation P_h of a hospital h is formed by inserting into P_h preferences over **infeasible sets**, sets containing multiple contracts between h and some doctor. This process will allow us to apply our results for many-to-many matching with contracts in the setting of many-to-one matching with contracts when preferences are completable in such a way that the completed preferences are substitutable.

Definition 11. A **completion** of a many-to-one preference relation P_f is a preference relation $\bar{P}_f$ satisfying the following conditions:

1. For $X', X'' \subseteq X$ (with $X'|_f = X'$ and $X''|_f = X''$), if $X' \succ_f X'' \succ_f \emptyset$ under P_f, then $X' \succ_f X'' \succ_f \emptyset$ under $\bar{P}_f$.

2. If $X' \subseteq X$ (with $X'|_f = X'$) is not individually rational for f under P_f, and if $|X'| = |X'_D|$, then X' is not individually rational for f under $\bar{P}_f$.[36]

Note that every preference relation is a completion of itself. Moreover, the defining conditions imply that for any doctor d, the preference profile P_d is the unique completion of P_d, as all doctors find only singleton sets individually rational.[37] Nonetheless, it is generally the case that nontrivial completions of hospital preferences exist. For example, consider the preference relation

$$P_h : \{x', z\} \succ \{x'\} \succ \{x\} \succ \{z\} \succ \varnothing$$

where $x'_D = x_D \neq z_D$. This preference relation admits the nontrivial completions

$$\bar{P}_h : \{x', x\} \succ \{x', z\} \succ \{x'\} \succ \{x\} \succ \{z\} \succ \varnothing$$
$$\hat{P}_h : \{x', z\} \succ \{x', x\} \succ \{x'\} \succ \{x\} \succ \{z\} \succ \varnothing.$$

Abusing terminology slightly, we say that a preference profile $\bar{P}$ is a **completion** of a preference profile P if, for each agent $f \in F$, the preference relation $\bar{P}_f$ in $\bar{P}$ is a completion of the associated preference relation P_f. Since a completion $\bar{P}$ of P differs from P only by the addition of preferences over infeasible sets, the following result is immediate.

Lemma 16. *Let $\bar{P}$ be a completion of a preference profile P. Then, an allocation $A \subseteq X$ is stable with respect to P if it is stable with respect to $\bar{P}$.*[38]

If there exists a substitutable completion of a preference profile P, then we say that P is **substitutably completable**. Here, a completion of a many-to-one preference profile P is substitutable whenever each completed preference relation $\bar{P}_h$ of a

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[36] Note that $X' \subseteq X$ is feasible in a many-to-one allocation if and only if $|X'| = |X'_D|$.

[37] If $f \in D$, then the second defining condition implies that any $X' \subseteq X$ (with $X'|_f = X'$) not individually rational for f under P_f is individually rational for f under $\bar{P}_f$, since $X'|_f = X'$ implies that $|X'| = |X'_D| = 1$ when $f \in D$. The equality of P_f and $\bar{P}_f$ then follows from the first defining condition.

[38] Recall that for many-to-one matching, strong stability is equivalent to stability. So the discussion of this section is still true when we consider strongly stable, instead of just stable, allocations.

hospital h is substitutable.[39] Viewing the completion of a preference profile as a preference profile in the setting of many-to-many matching with contracts, we may combine Lemma 16 with Theorem 10. This immediately proves that substitutable completability of preferences is sufficient to guarantee the existence of stable allocations—if $\bar{P}$ is a substitutable completion of a preference profile P, then there exists at least one allocation stable under P.

We now illustrate this discussion via extension of our earlier example. Consider the preference profile

$$\begin{aligned} P_h &: \{x', z\} \succ \{x'\} \succ \{x\} \succ \{z\} \succ \varnothing \\ P_{x_D} &: x \succ x' \\ P_{z_D} &: z. \end{aligned}$$

These preferences correspond to a setting in which h wants each of two tasks accomplished, x_D can accomplish both tasks, and z_D can only accomplish the latter task and cannot do so as well as x_D can.[40] As written, the preferences of h are not substitutable: hospital h rejects z when $\{x, z\}$ is available, but not when $\{x', x, z\}$ is available. However, it is possible to complete the preferences of h substitutably. Indeed, the preference relation

$$\bar{P}_h : \{x', x\} \succ \{x', z\} \succ \{x'\} \succ \{x\} \succ \{z\} \succ \varnothing$$

is a substitutable completion of P_h. One consequence of this example is that, like many-to-many matching more generally, completion allows us to state more "reasonable" preferences for hospitals. Indeed, by accounting for the fact that hospital h would most prefer x_D to accomplish both tasks if possible, we see that a substitutable preference relation underlies P_h. Furthermore, since P is substitutably completable, we know that there exists at least one allocation stable under P.

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[39] Note that since every preference profile is a completion of itself, all substitutable preference profiles are trivially substitutably completable.

[40] These preferences are very natural in the NRMP context. It may be that the former task is a research position, while the latter task is a clinical position in the same specialty.

There are, in fact, two stable allocations for P and the preferences of the doctors presented: $\{x\}$ and $\{x', z\}$, and this set of stable matches does not form a lattice in the usual way. Note that under $\bar{P}$ and the preferences of the doctors presented, there is only one stable match, $\{x\}$; although trivial, this single-element set of allocations is a lattice. The next section explores further the structure of the set of stable matches after preferences have been substitutably completed.

7.1 Completion and Lattice Structure

Substitutable completion in a sense restores the lattice structure of stable allocations observed by Hatfield and Milgrom (2005) for the narrower class of substitutable preference profiles. Specifically, for any substitutable completion of a preference profile, we obtain a lattice of stable matchings.

Theorem 17. *Suppose that $\bar{P}$ is a substitutable completion of a preference profile P. Then, the allocations stable with respect to $\bar{P}$ form a lattice. In particular, the conclusions of Corollary 11 apply.*

Although it is clear that any allocation stable under a completion of P must be stable under P, different completions of P may yield different sets of stable allocations. For example, consider the following nonsubstitutable preference profile P:

$$P_h : \{x, y'\} \succ \{x', y\} \succ \{x', y'\} \succ \{x, y\} \succ \{x'\} \succ \{y'\} \succ \{x\} \succ \{y\}$$

$$P_{x_D} : x \succ x'$$

$$P_{y_D} : y \succ y'.$$

There are three allocations stable under P: $\{x, y'\}$, $\{x', y\}$, and $\{x, y\}$. Additionally, there are two different possible substitutable completions of P_h:

$$\bar{P}_h : \{y, y'\} \succ \{x, y'\} \succ \{x', y\} \succ \{x', y'\} \succ \{x, y\} \succ \{x'\} \succ \{y'\} \succ \{x\} \succ \{y\},$$

$$\bar{\bar{P}}_h : \{x, x'\} \succ \{x, y'\} \succ \{x', y\} \succ \{x', y'\} \succ \{x, y\} \succ \{x'\} \succ \{y'\} \succ \{x\} \succ \{y\}.$$

The completed preference profiles induced by these completions of P_h yield different sets of stable allocations: $\{x', y\}$ and $\{x, y\}$ are stable under the first, while $\{x, y'\}$ and $\{x, y\}$ are stable under the second. Nonetheless, as Theorem 17 indicates, in each case the set of stable allocations forms a lattice.

7.2 Completion and the Law of Aggregate Demand

In addition to lattice structure, completion allows us to identify a class of nonsubstitutable many-to-one preference profiles to which the rural hospitals and strategy-proofness results of Hatfield and Milgrom (2005) extend.[41]

Theorem 18. *Suppose that there exists a substitutable completion $\bar{P}$ of a preference profile P such that all preference relations in $\bar{P}$ satisfy the law of aggregate demand. Then, the following results hold.*

1. *Each agent signs the same number of contracts at each allocation stable with respect to $\bar{P}$.*
2. *Suppose that a matching mechanism proceeds by first deterministically fixing a substitutable completion $\bar{P}$ of P such that all preference relations in $\bar{P}$ satisfy the law of aggregate demand, and then choosing the doctor-optimal stable allocation with respect to $\bar{P}$. In this case, that mechanism is (group) strategy-proof for doctors.*

The proof of the first part of Theorem 18 follows from application of Theorem 13. Likewise, the proof of the second part of Theorem 18 follows from direct extension of the method of Hatfield and Kojima (2009).

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[41] The class of such preference profiles is apparent from the conditions of Theorem 18; the example presented in the beginning of Section 7 presents one such preference profile.

7.3 Completion and Bilateral Substitutes

Hatfield and Kojima (2010) introduced the bilateral substitutes condition, which is weaker than the substitutes condition but is nonetheless sufficient to guarantee the existence of stable many-to-one contract allocations. Here, we compare the bilateral substitutes condition to that of substitutable completability, showing that neither condition is weaker than the other.

First, we recall the formal statement of the bilateral substitutes condition.

Definition 12. Contracts are **bilateral substitutes** for $h \in H$ if there do not exist contracts $x, z \in X$ and $Y \subseteq X$ such that $x_D, z_D \notin Y_D$, $z \notin C_h(Y \cup \{z\})$, and $z \in C_h(Y \cup \{x, z\})$.

All the substitutably completable preference relations we have discussed so far do in fact satisfy the bilateral substitutes condition. However, there are substitutably completable preference relations which do not satisfy bilateral substitutes. For example, consider the hospital preference relation

$$P_h : \{x, y, z\} \succ \{\hat{y}\} \succ \{x, y\} \succ \{x, z\} \succ \{y, z\} \succ \{y\} \succ \{x\} \succ \{z\}.$$

This relation may be substitutably completed by the addition of preferences over $\{y, \hat{y}\}$ to yield the relation

$$\bar{P}_h : \{y, \hat{y}\} \succ \{x, y, z\} \succ \{\hat{y}\} \succ \{x, y\} \succ \{x, z\} \succ \{y, z\} \succ \{y\} \succ \{x\} \succ \{z\}.$$

This example demonstrates that bilateral substitutes does not imply completablity. Thus, we see that substitutable completability is truly a "new" sufficient condition for the existence of stable many-to-one allocations—it includes a class of preference profiles which were not previously known to have stable allocations guaranteed.

Interestingly, however, substitutable completability is not a strictly weaker condition than bilateral substitutes. Indeed, there are preference relations such as

$$P_h : \{v, w\} \succ \{\hat{v}\} \succ \{\hat{w}\} \succ \{v\} \succ \{w\}$$

which satisfy bilateral substitutes but are not substitutably completable.

The notion of completion does, however, yield a sufficient condition for stability strictly weaker than bilateral substitutes. Specifically, combining Lemma 16 and the results of Hatfield and Kojima (2010) directly shows the following result.

Corollary 19. *Suppose that a preference profile* P *has a completion* $\bar{P}$ *in which all contracts are bilateral substitutes for their associated hospitals. Then, at least one allocation stable under* P *exists.*

This "bilaterally substitutable completability" condition is strictly weaker than the requirement of bilateral substitutes—to see this, it suffices to observe that the combination of the two examples presented in this section yields a preference relation which does not satisfy bilateral substitutes but has a completion which does. This condition is therefore the weakest known sufficient condition for the existence of stable many-to-one contract allocations, but it seems substantially less natural than substitutable completability. Moreover, it is not clear whether it is necessary.

8 Conclusion

Many-to-many matching with contracts is a general framework that can be used to describe buyer-seller markets with heterogeneous goods, labor market equilibria between firms and workers, allocation of consulting work between firms and consultants, and a variety of other important economic phenomena. In this framework, we have shown that substitutable preferences are necessary and sufficient for the existence of stable allocations. Furthermore, in the presence of substitutable preferences, the standard results regarding lattice structure, opposition of interests, the rural hospitals theorem, and Nash implementability continue to hold. Additionally, our results extend our understanding of many-to-one matching with contracts through the notion of substitutable completability, a new sufficient condition for the existence of stable many-to-one contract allocations.

Throughout, we have assumed that the market designer has complete control of the scope of possible contract language, but no power to prevent "blocks" that arise when parties deviate by recontracting within the provided language. In this setting, the stable allocations essential for applications of matching[42] are obtained only up to blocking deviations using contracts within the available language. This is admittedly a limiting assumption, as in practice agents who circumvent centralized clearinghouses contract outside of (and typically before) the matching mechanism.[43] However, when a matching mechanism's language is well-structured, so that reasonable, stable matches are obtained, agents should be glad to participate in the centralized match, and will not block the match by contracting outside of the designated language.

Our results imply that careful selection of the contract language is essential for functioning matching markets. Contract design can determine which—and more importantly, if—stable relationships can be found. Moreover, when the language is chosen effectively, many key results of matching theory apply. Selection of the optimal contract language depends upon application-specific parameters which the market designer must assess; hence, our work leaves substantial room for market design.

While this work proves a large number of key results for the theory of many-to-many matching, it also provides directions for future research. First, for the problem of matching couples to hospitals with multiple positions, we now know that a stable match is only theoretically guaranteed if both hospitals' and couples' preferences are substitutable. However, although one would not expect couples' preferences to be substitutable for practical applications such as the NRMP, stable couples matches appear to exist in practice (see Roth (2008)). Since it is now clear that substitutability

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[42] Roth (1984a), Roth (1991), and Roth and Xing (1994) provide empirical evidence that the stability of the outcome recommended by a centralized match is essential to the long-run success of the matching system.

[43] Clearly, there is no *a priori* reason why those agents should deal within the match's contract language.

is a necessary condition for stability, the infrequency of instabilities in the NRMP is puzzling.[44] Second, although we have found new sufficient conditions for the existence of a stable match in the many-to-one matching with contracts framework, no necessary and sufficient condition for this problem is known. Finally, although we have identified and examined tradeoffs in the design of contract languages, it is neither clear when languages induce substitutable preferences (and hence induce stability), nor how a putative language should be judged in practice. We leave these questions for future research.

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[44] Recent work of Kojima et al. (2010) and Ashlagi et al. (2011) argues that large-market effects may explain this phenomenon.

Appendix

Proof of Theorem 1

Suppose that the preferences of f are not matching-theory substitutable. Then there exist contracts $x, z \in X$ and $Y \subseteq X$ such that

$$z \notin C_f\left(Y \cup \{z\}\right) \text{ and } z \in C_f\left(\{x\} \cup Y \cup \{z\}\right).$$

Now consider any indirect utility function U which represents these preferences. Clearly, $U\left(Y\right) = U\left(Y \cup \{z\}\right)$, and so

$$U\left(Y \cup \{z\}\right) - U\left(Y\right) = 0 < U\left(\{x\} \cup Y \cup \{z\}\right) - U\left(\{x\} \cup Y\right)$$

and hence U is not submodular.

Suppose that the preferences of f are matching-theory substitutable. Suppose there are N sets of contracts that are individually rational for f, and that the preferences of f are given by

$$Pf : Z^N \succ Z^{N-1} \succ \cdots \succ Z^2 \succ Z^1 \succ \varnothing.$$

Let $U\left(Z^n\right) = 1 - 2^{-n}$.[45] Now consider any $Z \subseteq Y \subseteq X$ and $x \in X$. If $x \in Y$, then $C_f\left(Y\right) = C_f\left(\{x\} \cup Y\right)$ and we are done; if $x \notin C_f(\{x\} \cup Y)$, then the same conclusion holds. Now, if $x \notin Y$ and $x \in C_f\left(\{x\} \cup Y\right)$, then by matching-theory substitutability $x \in C_f\left(\{x\} \cup Z\right)$. Let $Z^n = C_f\left(Z\right)$ and $Z^{n'} = C_f\left(Y\right)$ where $n \leq n'$ as $Z \subseteq Y$. Hence $U\left(\{x\} \cup Z\right) - U\left(Z\right) \geq 2^{-n-1} \geq 2^{-n'-1} \geq U\left(\{x\} \cup Y\right) - U\left(Y\right)$ and so

$$U\left(\{x\} \cup Z\right) - U\left(Z\right) \geq U\left(\{x\} \cup Y\right) - U\left(Y\right)$$

and U is submodular by defintion.

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[45] A similar method is used by Chambers and Echenique (2009) to prove that for any increasing quasisupermodular function, there exists a monotonic transformation such that the transformed function is supermodular.

Proof of Theorem 2

Suppose the valuation function of f is auction-theory substitutable. Let $U(Y) = \pi(\tilde{p}(Y))$, where $\tilde{p}_j(Y) = \min_{p_j}\{(j, p_j) \in Y\}$ and $\pi(p)$ is the indirect profit function of f. Then, by Lemma 1 of Milgrom (2009) the indirect profit function $\pi(p)$ is submodular, and hence $U(Y)$ is submodular.

Conversely, suppose that the valuation function of f is not auction-theory substitutable. Then there exists p, p' such that $D(p)$ is a singleton, $j \in D(p)$, $p' \geq p$, and $p_j = p'_j$, but $j \notin D(p')$. Then $(j, p_j) \in C_f(O(p))$, but $(j, p_j) \notin C_f(O(p'))$ and $O(p') \subseteq O(p)$. Hence, the preferences of f are not matching-theory substitutable and, by Theorem 1, are not substitutable.

Proof of Theorem 3

Suppose that Y' is blocked in X' by some set of contracts $Z' \subseteq X'$.[46] Since $X \triangleright X'$, we have $Z' \subseteq X' \subseteq X$. But $Z' \not\subseteq Y'$ and by construction we must have $Z'|_f \subseteq C_f^X(Z' \cup Y')$ for each $f \in F$, contradicting the stability of Y with respect to X.

The proof for the case of strong stability is analogous.

Proof of Theorem 4

Let $Y' \equiv \{\Pi_d \cap \Lambda \cap \Pi_h \mid (d, h) \in D \times H\} \subseteq X'$, and note that Y' expresses Λ in X'. Now, suppose that Y' is blocked by some set of contracts $Z' \subseteq X'$.[47] By definition of a blocking set, we must have $Z' \not\subseteq Y'$; it follows that $Z \equiv Z' \cap (X - Y) \neq \emptyset$. We let $\hat{Y} \subseteq Y$ be a subset of Y primitive-equivalent to $Z' - Z$.[48] By construction, $(\hat{Y} \cup Z) \subseteq X$, $(\hat{Y} \cup Z) \not\subseteq Y$, and $(\hat{Y} \cup Z)|_f \subseteq C_f^X\left((\hat{Y} \cup Z) \cup Y\right)$, contradicting the stability of Y in X.

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[46] It is clear that Y' is individually rational, so Y' can only be unstable if it is blocked.

[47] As in the proof of Theorem 3, it is clear that Y' is individually rational, so Y' can only be unstable if it is blocked.

[48] It is clear that this set exists, since $Z' - Z \subseteq Y'$, which is primitive-equivalent to Y.

Proof of Theorem 5

If $P_f^{X'}$ is not substitutable for some $f \in F$, then there exist $z, x \in X'$ and $Y \subseteq X'$ such that $z \notin C_f^{X'}(Y \cup \{z\})$ but $z \in C_f^{X'}(Y \cup \{z, x\})$. But $P_f^{X'}$ is just the restriction of P_f^X to sets of contracts wholly contained in X', so in particular $z, x \in X'$ and $Y \subseteq X'$ comprise a counterexample to the substitutability of P_f^X.

The proof for the case of strong substitutability is analogous.

Proof of Lemma 6

Suppose that Y is a fixed point of Φ, let $\hat{Y} = C_D(Y)$, and let $Z = C_h(X - R_D(Y)) \neq \hat{Y}|_h$. If $Z \not\supseteq \hat{Y}|_h$, then there exists $y \in \hat{Y}|_h \cap R_H(X - R_D(Y))$, so $y \notin \Phi(Y)$ and Y is not a fixed point of Φ. If $Z \supsetneq \hat{Y}|_h$, there exists $z \notin \hat{Y}|_h$, such that $z \in C_h(X - R_D(Y))$, hence $z \in \Phi(Y) = Y$. But $z \notin \hat{Y}|_h \in C_D(Y)$, so $z \in R_D(Y)$, and hence $z \notin X - R_D(Y)$. But $z \in C_h(X - R_D(Y))$, a contradiction.

Proof of Lemma 7

If Z is a blocking set, then $Z \subseteq C_H(Y \cup Z)$ and $Z \subseteq C_D(Y \cup Z)$. Consider $z \in Z$; as the preferences of z_H and z_D are substitutable, $z \in C_{z_H}(Y \cup \{z\}) \cap C_{z_D}(Y \cup \{z\})$. Hence $\{z\}$ is a blocking set for Y.

Proof of Theorem 8

If $\hat{Y} \equiv C_D(Y)$ is not stable, then there are three possibilities, all of which we proceed to rule out:

1. $\hat{Y}|_d$ is not individually rational for some $d \in D$. But then $\hat{Y}|_d \not\subseteq C_d(Y)$, contradicting the assumption that $\hat{Y} \equiv C_D(Y)$.
2. There exists a hospital h such that $\hat{Y}|_h$ is not individually rational for h. Then, since $\hat{Y}|_h \subseteq C_D(Y)$, there is at least one set of contracts in $X - R_D(Y)$ that is

strictly preferred by h to $\hat{Y}|_h$. Then $C_h\left(X - R_D(Y)\right) \neq \hat{Y}|_h$ and we are done by Lemma 6.

3. There exists a set Z that blocks $\hat{Y}$. Then for $z \in Z$, $\{z\}$ blocks $\hat{Y}$ by Lemma 7. Hence $z \notin Y$ (as otherwise, $C_D(Y) \neq \hat{Y}$) so $z \in X - R_D(Y)$. But we also know that $z \in C_{z_H}\left(\hat{Y} \cup \{z\}\right)$, so $C_{z_H}\left(X - R_D(Y)\right) \neq \hat{Y}|_{z_H}$ and we are done by Lemma 6.

Proof of Theorem 9

Let Y be the largest set of contracts such that $C_D(Y) = \hat{Y}$. We first show such a set Y exists. First, $C_D\left(\hat{Y}\right) = \hat{Y}$ as $\hat{Y}$ is individually rational for doctors, so there does exist a set Y such that $C_D(Y) = \hat{Y}$. Now, suppose that

$$\hat{Y} = C_D\left(\hat{Y} \cup Z^i\right) \text{ for } i = 1, 2$$

and that $\hat{Y} \neq C_D\left(\hat{Y} \cup Z^1 \cup Z^2\right)$. Then there exists $z \in C_D\left(\hat{Y} \cup Z^1 \cup Z^2\right)$ such that $z \notin \hat{Y}$. But then $z \in Z^i$ for some i, and by substitutability $z \in C_D\left(\hat{Y} \cup Z^i\right)$, violating our earlier assumption that $\hat{Y} = C_D\left(\hat{Y} \cup Z^i\right)$.

Now, $C_D(Y) = \hat{Y}$ and for any $z \notin Y$, $z \in C_D\left(Y \cup \{z\}\right) \neq \hat{Y}$. Hence,

$$X - R_D(Y) = (X - Y) \cup \hat{Y} \equiv W.$$

If $C_H(W) = \hat{Y}$ we are done, as

$$X - \left(W - \hat{Y}\right) = X - \left(\left((X - Y) \cup \hat{Y}\right) - \hat{Y}\right) = X - (X - Y) = Y.$$

If not, then suppose $C_H(W) = Z \neq \hat{Y}$. If $Z \subsetneq \hat{Y}$, then $C_H\left(\hat{Y}\right) = Z$, and so $\hat{Y}$ is not individually rational for hospitals. If $Z \not\subseteq \hat{Y}$, then there exists $z \in Z = C_H(W)$ such that $z \notin \hat{Y}$, hence subsitutability implies that $z \in C_H\left(\hat{Y} \cup \{z\}\right)$. Moreover, we know that $z \in C_D\left(Y \cup \{z\}\right)$, as Y is the largest set such that $C_D(Y) = \hat{Y}$. Hence $\{z\}$ is a blocking set for $\hat{Y}$, a contradiction.

Proof of Corollary 11

Let $\hat{Y}$ be the fixed point of Φ such that $C_D(\hat{Y}) = Y$, and let $\hat{Y}'$ be the fixed point of Φ such that $C_D(\hat{Y}') = Y'$. Then since the doctors uniformly prefer Y, we have that $\hat{Y} \supseteq \hat{Y}'$. Hence $X - R_D\left(\hat{Y}\right) \subseteq X - R_D\left(\hat{Y}'\right)$. We know from Lemma 6 that the hospitals attain $C_H\left(X - R_D\left(\hat{Y}\right)\right)$ in the stable match Y and $C_H\left(X - R_D\left(\hat{Y}'\right)\right)$ in the stable match Y'. It follows that all hospitals prefer Y' to Y.

The second statement of the corollary follows from the first.

Proof of Theorem 12

If the preferences of a hospital h are not substitutable, then there exist contracts $x, z \in X|_h$ and a set of contracts $Y \subseteq X - \{x, z\}$ such that $Y_H = \{h\}$ and

$$\begin{aligned} &z \notin C_h(Y \cup \{z\}) \\ &z \in C_h(\{x\} \cup Y \cup \{z\}). \end{aligned}$$

There are two cases to consider.

Case 1: $x_D \neq z_D$. By assumption, there must exist a hospital $h' \neq h$. Furthermore, there must exist contracts x' and z' with $x_D = x'_D$, $z_D = z'_D$ and $x'_H = z'_H = h'$. Let z_D have preferences such that

$$C_{z_D}(W) = \begin{cases} (W \cap (Y \cup \{z\}))|_{z_D} & \{z, z'\} \subseteq W \\ (W \cap (Y \cup \{z, z'\}))|_{z_D} & \text{otherwise.} \end{cases}$$

That is, z_D is willing to accept any and all of the contracts he is associated with in Y, and z_D wants one of z and z', preferring z, and rejects all other contracts. Let x_D have preferences such that

$$C_{x_D}(W) = \begin{cases} (W \cap (Y \cup \{x'\}))|_{x_D} & \{x, x'\} \subseteq W \\ (W \cap (\{x\} \cup Y \cup \{x'\}))|_{x_D} & \text{otherwise.} \end{cases}$$

Let h' have preferences such that

$$C_{h'}(W) = \begin{cases} (W \cap (Y \cup \{z'\}))|_{h'} & \{z', x'\} \subseteq W \\ (W \cap (Y \cup \{z', x'\}))|_{h'} & \text{otherwise.} \end{cases}$$

Let every doctor $d \in D - \{x_D, z_D\}$ have preferences such that

$$C_d(W) = (W \cap Y)|_d.$$

Consider any allocation A; we will show A can not be stable.

1. Suppose $A|_h \prec_h C_h(Y \cup \{z\})$. If A is individually rational for all hospitals, then $C_h(Y \cup \{z\})$ blocks A, as all doctors choose their contracts in $C_h(Y)$.
2. Suppose $A|_h = C_h(Y \cup \{z\})$. Then $z' \in A$, as otherwise $\{z'\}$ blocks A. But then $C_h(\{x\} \cup Y \cup \{z\})$ blocks A.
3. Suppose $C_h(\{x\} \cup Y \cup \{z\}) \succ_h A|_h \succ_h C_h(Y \cup \{z\})$. In this case, if A is individually rational for all hospitals, then $A \subseteq \{x, x', z'\} \cup Y \cup \{z\}$; then $x \in A$ as otherwise we could not have $A|_h \succ_h C_h(Y \cup \{z\})$. But then, $C_h(\{x\} \cup Y \cup \{z\})$ blocks A.
4. Suppose $C_h(\{x\} \cup Y \cup \{z\}) = A|_h$. Then if $z' \in A$, the allocation A is not individually rational for z_D, and if $x' \in A$, the allocation A is not individually rational for x_D; but this implies that $\{x'\}$ blocks A.

Case 2: $x_D = z_D \equiv d$. By assumption, there are two hospitals, h' and h'', such that $h \neq h' \neq h'' \neq h$ and one doctor $\hat{d} \neq d$. Now consider the contracts $x', x'', \hat{x}'$, and $\hat{x}''$ such that $x'_D = x''_D = d$, $\hat{x}'_D = \hat{x}''_D = \hat{d}$, $x'_H = \hat{x}'_H = h'$ and $x''_H = \hat{x}''_H = h''$, which exist by assumption. Let d have preferences such that

$$C_d(W) = (W \cap Y)|_d \cup \tilde{C}_d(W \cap \{x, z, x', x''\})$$

where $\tilde{C}_d(\tilde{W})$ is the responsive choice function over $\{x, z, x', x''\}$ with quota 2 and underlying preference order $x'' \succ z \succ x \succ x'$. Let $\hat{d}, h'$, and h'' have preferences such that

$$C_{\hat{d}}(W) = \begin{cases} (W \cap (Y \cup \{\hat{x}'\}))|_{\hat{d}} & \{\hat{x}', \hat{x}''\} \subseteq W \\ (W \cap (Y \cup Z \cup \{\hat{x}', \hat{x}''\}))|_{\hat{d}} & \text{otherwise,} \end{cases}$$

$$C_{h'}(W) = \begin{cases} (W \cap (Y \cup \{x'\}))|_{h'} & \{x', \hat{x}'\} \subseteq W \\ (W \cap (Y \cup \{x', \hat{x}'\}))|_{h'} & \text{otherwise,} \end{cases}$$

$$C_{h''}(W) = \begin{cases} (W \cap (Y \cup \{\hat{x}''\}))|_{h} & \{\hat{x}'', x''\} \subseteq W \\ (W \cap (Y \cup \{\hat{x}'', x''\}))|_{h} & \text{otherwise.} \end{cases}$$

Finally, let every doctor $\bar{d} \in D - \{d, \hat{d}\}$ have preferences such that

$$C_{\bar{d}}(W) = (W \cap Y)|_{\bar{d}}.$$

Consider any allocation A; we will show that A can not be stable.

1. Suppose $A|_h \prec_h C_h(Y \cup \{z\})$. Then $C_h(Y \cup \{z\})$ blocks A, as all the doctors choose their contracts in $C_h(Y \cup \{z\})$.
2. Suppose $A|_h = C_h(Y \cup \{z\})$. Since d does not obtain x or z, he desires both x' and x''. If A is stable then, $x' \in A$. Furthermore, since $\hat{d}$ does not obtain $\hat{x}'$, for A to be stable, we must have $\hat{x}'' \in A$. Hence, if A is stable, $\{\hat{x}'', x'\} \subseteq A$ and $x'' \notin A$. In that case, $C_h(\{x\} \cup Y \cup \{z\})$ blocks A.
3. Suppose $C_h(Y \cup \{z\}) \prec_h A|_h \prec C_h(\{x\} \cup Y \cup \{z\})$. Then $x \in A$, so $C_h(\{x\} \cup Y \cup \{z\}) - \{x\}$ blocks A, as d will always choose z and the other doctors in Y will always accept offers of any and all contracts in Y.
4. Suppose $C_h(\{x\} \cup Y \cup \{z\}) = A|_h$. If $\hat{x}' \notin A$, then $\{\hat{x}'\}$ blocks A. (Note that if $x' \in A$, then $\{x, z, x'\} \subseteq A$, and so A is not individually rational for d.) But $\hat{x}' \in A$ implies that $\hat{x}'' \notin A$. Hence $\{x''\}$ blocks A. (Note that $x'' \notin A$, as then $\{x'', x, z\} \subseteq A$, and so A is not individually rational for d.)

Proof of Theorem 13

It suffices to show that Consider any stable allocation A, and the doctor-optimal stable allocation A^*. Since every hospital prefers A to A^* from Corollary 11, we know from the law of aggregate demand that the number of contracts signed by each hospital is weakly smaller at A^*, hence $|A^*| \leq |A|$. Hence if any doctor receives strictly more contracts at A^* than at A, some doctor must receive strictly fewer contracts at A^* than at A. This cannot happen, as every doctor is weakly better off at A^* than at A, and every doctor's preferences satisfy the law of aggregate demand. Thus every doctor receives the same number of contracts at every stable allocation.

An analogous argument shows the result for hospitals.

Proof of Theorem 15

The forwards direction is trivial, hence we show only the reverse direction. Without loss of generality, we assume that all hospital preferences are strongly stable. Now, we fix preferences, and consider any stable allocation A. If A is not strongly stable, then there exists a set Z such that for each $f \in Z_F$ there exists an individually rational Y^f such that $Z|_f \subseteq Y^f \subseteq Z \cup A$ and $Y^f \succ_f A$. Now, consider a doctor $d \in Z_D$. Since $Y^d \succ_d A|_d$, $C_d(Z \cup A) \neq A|_d$ and hence, as $A|_d$ is individually rational for d, there exists $x \in C_d(Z \cup A)$ such that $x \in Z - A$. Hence, by substitutability, we have $x \in C_d(\{x\} \cup A)$. Now, if $x \in C_{x_H}(\{x\} \cup A)$, then $\{x\}$ blocks A, contradicting the stability of A. Hence, $x \notin C_{x_H}(\{x\} \cup A)$, but $x \in C_{x_H}(Y^{x_H})$ and

$$C_{x_H}(Y^{x_H}) = Y^{x_H}|_{x_H} \succ_{x_H} C_{x_H}(A) = C_{x_H}(\{x\} \cup A).$$

But $x \in [A \cup \{x\}] \cap C_{x_H}(Y^{x_H}) - C_{x_H}(A \cup \{x\})$, so the preferences of x_H are not strongly substitutable.

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