Mathematical Appendix

to

"On Derivatives Markets and Social Welfare"

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1 Basic Framework

There is a finite set of agents I. There is also a finite set of firms F, each of which has a unit measure of stock available; let the (possibly negative) fractional economic ownership of firm $f \in F$ held by agent $i \in I$ be denoted $s^i_f \in \mathbb{R}$.Hence, each agent has a *portfolio* $s^i \equiv (s^i_f)_{f\in F}$, and a portfolio profile is a vector of agent portfolios $(s^i)_{i\in I}$ such that

$$\sum_{i\in I} s^i_f = 1 \text{ for all } f \in F.$$

We denote the set of all portfolio profiles as $\mathcal{S} \equiv \{s \in \mathbb{R}^{I\times F} : \sum_{i\in I} s^i_f = 1 \text{ for all } f \in F\}$. We assume that agents are initially endowed with shares, that is, there is an initial endowment $(e^i_f)_{i\in I, f\in F}$ portfolio profile.

There is also a set of shareholder motions M. Each shareholder motion $m \in M$ represents a binary decision by a firm $\delta_m \in \{0, 1\}$. This decision may affect the value of that firm, and also may impact the values of other firms. For instance, a motion m may represent a firm's decision to enter a market; if entry occurs, then existing firms in that market face additional competition.

Each agent i submits a ballot $\beta^i_m \in [0, 1]$ on each motion m; we call the matrix β a ballot profile. Furthermore, for each $m \in M$ there exists a profile of control rights $(r^i_m)_{i\in I}$ such that $r^i_m \geq 0$ for all $i \in I$ and $\sum_{i\in I} r^i_m = 1$.[1] We say that m passes—$\delta_m(\beta) = 1$—if and only if $\sum_{i\in I} r^i_m \beta^i_m \geq \alpha_m$, where $\alpha_m \in [0, 1]$ is the minimum percentage of votes required for passage of motion m, and that m fails—$\delta_m(\beta) = 0$—otherwise. After voting, the value of firm f is given by $v_f(\delta(\beta))$.

An outcome $\langle s, \beta\rangle$ is a portfolio profile s and ballot profile β. The utility of agent i for an outcome $\langle s, \beta\rangle$ is given by

$$u^i(\langle s, \beta\rangle) \equiv \sum_{f\in F} s^i_f v_f(\delta(\beta)).$$

[1] For now, we assume that the distribution of control rights is fixed; we relax this assumption in Section 2.

An outcome $\langle s, \beta \rangle$ is *efficient* if its ballot profile leads to a decision which maximizes the total value of the economy, i.e.,

$$\beta \in \arg\max_{\hat{\beta} \in [0,1]^{I \times M}} \sum_{i \in I} \sum_{f \in F} s_f^i v_f(\delta(\hat{\beta})) = \arg\max_{\hat{\beta} \in [0,1]^{I \times M}} \sum_{f \in F} v_f(\delta(\hat{\beta})).$$

We let the column vector $(p_f)_{f \in F}$ denote the prices at which each of the firms trades initially. An arrangement $[e; \langle s, \beta \rangle; p]$ is an initial endowment e, an outcome $\langle s, \beta \rangle$, and a price vector p.

Given an initial endowment e, an arrangement $[e; \langle s, \beta \rangle; p]$ induces a utility for agent i of

$$\tilde{u}^i([e; \langle s, \beta \rangle; p]) \equiv \sum_{f \in F} \left(s_f^i v_f(\delta(\beta)) - p_f(s_f^i - e_f^i) \right).$$

An arrangement $[e; \langle s, \beta \rangle; p]$ is efficient if the associated outcome $\langle s, \beta \rangle$ is efficient.

1.1 Competitive Equilibrium

The demand correspondence $D^i(e; \beta; p)$ for agent i, given the ballot matrix $\beta \in [0,1]^{I \times M}$, is given by

$$D^i(e; \beta; p) \equiv \arg\max_{s \in \mathcal{S}} \tilde{u}^i([e; \langle s, \beta \rangle; p])$$

and the demand correspondence for the entire economy, $D(e; \beta; p)$, is given by

$$D(e; \beta; p) \equiv \bigcap_{i \in I} D^i(e; \beta; p).$$

We now define the concept of *competitive equilibrium*.

Definition 1. A *competitive equilibrium*, given an initial endowment e, is an arrangement $[e; \langle s, \beta \rangle; p]$ such that

1. $s \in D(e; \beta; p)$;

2. β is a Nash equilibrium in undominated strategies of the induced voting game;[2,3]

3. there is no agent i and $(\hat{s}^i, \hat{\beta}^i)$ such that

$$\tilde{u}^i([e; \langle(\hat{s}^i, s^{-i}), (\hat{\beta}^i, \beta^{-i})\rangle; p]) > \tilde{u}^i([e; \langle s, \beta\rangle; p]).$$

Condition 1 states that the demand correspondence is non-empty; that is, each agent is demanding an optimal portfolio given the equilibrium voting behavior. Furthermore, Condition 1 ensures that markets clear, as the demand correspondence is only non-empty if supply of stock for each firm f equals demand. Condition 2 ensures that each agent is voting optimally, given his stock holdings and the ballot profile of other agents. Finally, Condition 3 states that no agent can strictly increase his utility by both changing his portfolio and his ballot.[4] Formally, Condition 1 follows from Condition 3; however, we retain Condition 1 as it encapsulates the standard definition of competitive equilibrium.

1.1.1 Inefficiency

We show that competitive equilibria, when they exist, may be inefficient.

Example 1. Suppose there are two firms, $F = \{f, g\}$ and four agents $\{i, j, k, \ell\}$. There is one motion m which affects the value of both firms, and at least $\alpha_m = \frac{1}{2}$ of the votes are necessary for passage. Let

	$\delta_m = 0$	$\delta_m = 1$
$v_f(\delta_m)$	16	8
$v_g(\delta_m)$	6	21

.

[2]Technically, the Nash equilibrium under consideration may be mixed; in this case, the expression $\tilde{u}^i([e; \langle s, \beta\rangle; p])$ must be replaced by the analogous expression in terms of expected utilities. As agents are risk-neutral, this concern is immaterial to the analysis, and so we suppress it for ease of exposition.

[3]This condition ensures that each agent votes as to maximize the value of his portfolio given the voting strategies of the other agents. As is common in voting games, we impose the restriction that agents vote as-if-pivotal; see, e.g., Baron and Ferejohn (1989) and Austen-Smith and Banks (1996).

[4]Note that Condition 3, as in standard definitions of competitive equilibrium, allows for portfolio allocations that do not necessarily satisfy market clearing condition. The definition of an agent's utility as a function of his shareholding naturally extends to such cases.

Now let $r_m = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$, where we use the convention that vectors' coordinates are listed in alphabetical order. Let the endowment e be given by

$$e = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now show that $[e; \langle e, (0,0,0,1)\rangle; (16,6)^\top]$ is a competitive equilibrium. The first condition for competitive equilibrium is clearly satisfied, as the price of a share of either firm is equal to its value given the voting profile. Furthermore, each agent is voting so as to maximize the value of his shares; hence, the second condition for competitive equilibrium holds. Finally, note that no one agent changing his vote changes δ_m, and so the third condition for competitive equilibrium is vacuously satsified given that the first condition is.

However, the outcome $\langle e, (0,0,0,1)\rangle$ associated with this arrangement is not efficient, as

$$v_f(\delta((0,0,0,1))) + v_g(\delta((0,0,0,1))) = 22 < 29 = v_f(\delta((1,1,1,1))) + v_g(\delta((1,1,1,1))).$$

Intuitively, the competitive equilibrium in Example 1 is inefficient since each agent (correctly) believes that his vote will not be pivotal, and so long as each of i, j, and k holds his original portfolio, each of them is both holding an optimal portfolio and voting optimally. Hence the inefficient arrangement $[e; \langle e, (0,0,0,1)\rangle; (16,6)^\top]$ satisfies the definition of competitive equilibrium given in Definition 1. Switching to the more efficient voting outcome $\delta_m = 1$ would require coordination amongst the agents, and such coalitional deviations are not considered by the competitive equilibrium solution concept. However, this problem is not alleviated by allocating the control rights to a single agent; in that case, competitive equilibria may not exist at all, as we show in the next section.

1.1.2 Non-Existence

We now show that competitive equilibria may not exist.

Example 2. Let $F = \{f, g\}$ and $I = \{i, j\}$. There is one motion m which affects the value of both firms, and at least $\alpha_m = \frac{1}{2}$ of the votes are necessary for passage. Let

	$\delta_m = 0$	$\delta_m = 1$
$v_f(\delta_m)$	8	4
$v_g(\delta_m)$	2	4

.

Now let $r_m = (1, 0)$ where we again use the convention that vectors' coordinates are listed in alphabetical order. Let the endowment e be given by

$$e = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Now consider any arrangement of the form $[e; \langle s, \beta \rangle; p]$ where $\delta_m(\beta) = 0$. Then, in order for the market for shares to clear, we must have that $p = (8, 2)^\top$.[5] Now consider the following deviation by agent i. Let $(\hat{s}^i, \hat{\beta}^i) = ((0, \frac{1}{2}), (1))$. Note that i's utility under the original arrangement $[e; \langle s, \beta \rangle; p]$ is 5, regardless of s. However, his utility under the arrangement induced by the deviation $[e; \langle (\hat{s}^i, s^{-i}), (\hat{\beta}^i, \beta^{-i}) \rangle; p]$ is 6.

Now consider any arrangement of the form $[e; \langle s, \beta \rangle; p]$ where $\delta_m(\beta) = 1$. Then, in order for the market for shares to clear, we must have that $p = (4, 4)^\top$. Now consider the following deviation by agent i. Let $(\hat{s}^i, \hat{\beta}^i) = ((1, \frac{1}{2}), (0))$. Note that i's utility under the original arrangement $[e; \langle s, \beta \rangle; p]$ is 4, regardless of s. However, his utility under the arrangement induced by the deviation $[e; \langle (\hat{s}^i, s^{-i}), (\hat{\beta}^i, \beta^{-i}) \rangle; p]$ is 7.

This example shows that when the control rights are allocated to a single agent, competitive equilibria will typically fail to exist. The central difficulty is that the prices of the various firms must reflect a particular decision by the agent with the control right; other-

[5] For any other set of prices, the demand correspondence for each agent is empty, as each agent will have unbounded demand. For instance, if $p_f > 8$, each agent's demand for firm f is unbounded. (This same analysis applies throughout the paper when the price of a share of a firm does not reflect its final value.)

wise, any agent would be able to generate an arbitrarily large profit by buying or selling an arbitrarily large amount of shares of the firm whose price does not reflect the decision that it is believed will be made. However, under such a scenario, the prices can not reflect the value of the firm should some other decision be made. Hence, the agent with the control right can generate an arbitrarily large profit by both changing the decision which is made and buying or selling shares in order to profit from that decision.

1.2 Core-Compatibility

Definition 2. An outcome $\langle s, \beta \rangle$ is in the *core* if there does not exist a set of agents $J \subseteq I$, and $\hat{\beta}^J$ such that

$$\sum_{j \in J} u^j(\langle s, (\hat{\beta}^J, \beta^{-J}) \rangle) > \sum_{j \in J} u^j(\langle s, \beta \rangle). \tag{1}$$

The standard definition of the core is that there does not exist a coalition and an action for each member of that coalition such that each member is weakly better off with one member being strictly better off. This definition is equivalent to the standard definition: if a coalition J and voting profile $\hat{\beta}^J$ satisfying (1) existed, then they could implement transfers amongst each other and strictly increase the utility of each $j \in J$.

We say that an initial endowment e is *core-compatible* if, for every ballot profile $\hat{\beta}$, there exists a *core arrangement* $[e; \langle s, \beta \rangle; p]$ *associated with* e, such that the outcome $\langle s, \beta \rangle$ is in the core and, for all $i \in I$,

$$\tilde{u}^i([e; \langle s, \beta \rangle; p]) \geq \tilde{u}^i([e; \langle e, \hat{\beta} \rangle; p])$$

holds.[6] Intuitively, an initial endowment is core-compatible if agents can engage in mutually beneficial transactions so as to obtain a core outcome, irrespective of their belief about voting behavior.

[6] Note that we do not impose any restriction on beliefs about balloting, i.e., $\hat{\beta}$, except that agents' expectations about the balloting are consistent.

1.2.1 Efficiency

We first show that core outcomes maximize social welfare.

Theorem 1. *Core outcomes are efficient.*

Proof. Suppose that $\langle s, \beta \rangle$ is not efficient. Then there exists $\bar{\beta}$ such that

$$\sum_{i \in I} u^i(\langle s, \beta \rangle) < \sum_{i \in I} u^i(\langle s, \bar{\beta} \rangle).$$

But then taking $J = I$ and $\hat{\beta} = \bar{\beta}$, we see that (1) is not satisfied, and hence $\langle s, \beta \rangle$ is not in the core. □

In particular, Theorem 1 shows that if the initial endowment e is core-compatible, then agents should trade so that the voting outcome will be efficient.

Corollary 1. *For any core-compatible initial endowment e, the associated core arrangement $[e; \langle s, \beta \rangle; p]$ is efficient.*

1.2.2 Existence

We now show that for every initial endowment, there exists a mutually beneficial set of transactions which produces a core outcome. To do this, we first show that the core is nonempty and characterize a subset of the core.

Lemma 1. *The core is nonempty. In particular, every outcome $\langle s, \beta \rangle$ such that*

1. *for all $i, j \in I$, we have $s^i = a s^j$ for some $a \in \mathbb{R}_{\geq 0}$, and*

2. *β is efficient,*

is in the core.

Proof. Consider any such $\langle s, \beta \rangle$. To see that it satisfies the conditions of Definition 2, note that, for any deviating coalition $J \subseteq I$,

$$\sum_{j \in J} \sum_{f \in F} s_f^j v_f(\delta(\beta)) \propto \sum_{i \in I} \sum_{f \in F} s_f^i v_f(\delta(\beta)).$$

Hence, maximizing the welfare of any coalition is equivalent to maximizing the welfare of the grand coalition. But the ballot profile β maximizes welfare; hence, $\langle s, \beta \rangle$ is in the core. $\square$

We now show that the core can be reached from any initial endowment through mutually beneficial exchange.

Theorem 2. *Every initial endowment is core-compatible.*

Proof. Consider an initial endowment vector e. Consider some $\hat{\beta}$ and let $p_f = v_f(\delta(\hat{\beta}))$. The utility of agent i under the initial allocation and voting profile $\hat{\beta}$ is

$$\tilde{u}^i([e; \langle e, \hat{\beta} \rangle; p]) = \sum_{f \in F} e_f^i v_f(\delta(\hat{\beta})). \tag{2}$$

Let $\langle s, \beta \rangle$ be any outcome of the form given in Lemma 1. The utility of agent i under the arrangement $[e; \langle s, \beta \rangle; p]$ is

$$\begin{aligned} \tilde{u}^i([e; \langle s, \beta \rangle; p]) &= \sum_{f \in F} s_f^i v_f(\delta(\beta)) - p_f(s_f^i - e_f^i) \qquad (3) \\ &= \sum_{f \in F} s_f^i v_f(\delta(\beta)) - v_f(\delta(\hat{\beta}))(s_f^i - e_f^i) \\ &= \sum_{f \in F} s_f^i (v_f(\delta(\beta)) - v_f(\delta(\hat{\beta}))) + \sum_{f \in F} v_f(\delta(\hat{\beta})) e_f^i. \end{aligned}$$

Hence, the difference between utility under the arrangement $[e; \langle s, \beta \rangle; p]$, given by (3), and the utility under the arrangement $[e; \langle e, \hat{\beta} \rangle; p]$, given by (2), is given by

$$\sum_{f \in F} s_f^i (v_f(\delta(\beta)) - v_f(\delta(\hat{\beta})))$$

which is non-negative as $\delta(\beta)$ is efficient and $s^i_f = s^i_{f'} > 0$ for all $i \in I$ and $f, f' \in F$. Hence,

$$\tilde{u}^i([e; \langle s, \beta \rangle; p]) \geq \tilde{u}^i([e; \langle e, \hat{\beta} \rangle; p]).$$

and so $[e; \langle s, \beta \rangle; p]$ is core-compatible. □

To reach a core outcome, given any endowment, agents may trade stock, taking into account how such trades will change voting incentives. Once at a core outcome, no set of agents may improve their joint utility by changing their voting profile. This analysis illustrates the key distinction between the competitive equilibrium and core-compatibility solution concepts: the competitive equilibrium solution concept does not allow agents to consider how buying or selling stock will affect voting decisisons, while core-compatibility explicitly considers such incentives.

To illustrate how an economy with inefficient competitive equilibria progresses to the core, we return to the setting of Example 1. For this example, a core-compatible arrangement is given by

$$\left[\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \left\langle \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, (1,1,1,1) \right\rangle, \begin{pmatrix} 16 \\ 6 \end{pmatrix}\right].$$

To reach this outcome, agents i, j, and k each buy $\frac{1}{4}$ of a share of firm g from agent ℓ at a price consistent with a vote producing $\delta_m = 0$; agent ℓ is happy to engage in such a trade, as doing so changes the voting incentives of agents i, j, and k. Their changed votes increase the value of agent ℓ's remaining holdings. Similarly, agent ℓ buys $\frac{1}{12}$ of a share of firm f from each of i, j, and k. At this outcome, no agents may profitably deviate, as each agent holds an identical portfolio, and so voting to maximize the surplus of any subset of agents is equivalent to voting to maximize social surplus.[7] Similar analysis identifies equally

[7]Note that here, for expositional clarity, we have focused on a core outcome in which all agents hold market portfolios. This is not necessary in general—any core outcome can be supported in a core-compatible arrangement.

well-behaved core-compatible arrangements for Example 2.

Although we have heretofore treated each motion as an abstract object, which is not specifically linked to any firm, in practice a motion represents the decision by shareholders of a specific firm; this structure can be modeled in our setting by associating each motion $m \in M$ with a specific firm $\varphi(m) \in F$. With such an identification φ, an outcome $\langle s, \beta \rangle$ involves *empty voting* if there exists an agent $i \in I$ and a motion $m \in M$ such that

1. $r^i_m > 0$, and
2. $s^i_{\varphi(m)} \leq 0$.

The first condition states that the agent i has positive control rights over a particular shareholder motion m associated with the firm $\varphi(m)$, while the second condition states that i has non-positive economic interest in $\varphi(m)$. This notion is stronger than the definition of empty voting that has been discussed by Hu and Black (2005, 2007, 2008), who label "empty voting" any situation in which a shareholder's voting interest exceeds that shareholder's economic interest.

For instance, in the context of Example 1, we assume that $\varphi(m) = f$, and consider the arrangement

$$\left[\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \left\langle \begin{pmatrix} -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{11}{8} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}, (1,1,1,1) \right\rangle, \begin{pmatrix} 16 \\ 6 \end{pmatrix} \right].$$

In this arrangement, all three of i, j, and k engage in empty voting, as they are voting on the shareholder proposal m associated with f while each holding a negative amount of shares in firm f. Furthermore, it can be shown that this is a core-compatible arrangement, and hence this example demonstrates that socially efficient outcomes may be associated with empty voting.

1.3 Private Information

We extend our earlier model to incorporate the possibility of market uncertainty: There is a set of states of the world Ω, and the value of each firm f is now a function of both the shareholder vote as well as the true state of the world, $v_f(\delta;\omega)$. There exists a probability distribution π over states of the world; furthermore, after trading, each agent $i \in I$ receives a signal σ^i which depends on the underlying state.[8] The probability that $i \in I$ receives the signal $\sigma^i \in \Sigma^i$ given the state of the world $\omega \in \Omega$ is $n^i_{\sigma^i}(\omega)$.

We augment the model by allowing each agent i to send a public message μ^i after receiving his signal σ^i but before voting takes place; for simplicity, we assume that the message space available to agent i is simply Σ_i. An *outcome* $\langle s, (\mu^i(\sigma^i))_{i\in I}, (\beta^i(\mu,\sigma^i))_{i\in I}\rangle$ is a portfolio profile s, a message profile μ, and a voting profile β, where an agent's vote may now depend on his private signal as well as the profile of messages.

An outcome is *efficient* if the decision vector $\delta(\beta)$ induced by the message profile $(\mu^i(\sigma^i))_{i\in I}$ and voting profile $(\beta(\mu,\sigma^i))_{i\in I}$ maximizes the expected value of the economy conditional on the set of signals, i.e.,

$$\beta \in \operatorname*{arg\,max}_{\hat{\beta}\in[0,1]^{I\times M}} \mathbb{E}\left[\sum_{i\in I}\sum_{f\in F} s^i_f v_f(\delta(\hat{\beta})) \;\middle|\; \sigma\right] = \operatorname*{arg\,max}_{\hat{\beta}\in[0,1]^{I\times M}} \mathbb{E}\left[\sum_{f\in F} v_f(\delta(\hat{\beta})) \;\middle|\; \sigma\right].$$

for each $\sigma \in \times_{i\in I}\Sigma^i$.

Definition 3. An outcome $\langle s,\mu,\beta\rangle$ is in the *full revelation core* if

1. each agent reports truthfully, i.e., $\mu^i = \sigma^i$ for all $i \in I$, and

2. there does not exist a set of agents $J \subseteq I$ and signals $(\sigma^j)_{j\in J}$ for those agents such

[8] Note that we assume that no trade occurs after agents obtain information about the underlying security, as such a trade would reveal the information of any agent endeavoring to trade, thus making the information public: see Milgrom and Stokey (1982).

that there exists messages $(\hat{\mu})_{j \in J}$ and voting profile $\hat{\beta}$ such that

$$\mathbb{E}\left[\sum_{j \in J} u^j(\langle s, (\hat{\mu}^J, \mu^{-J}), (\hat{\beta}^J, \beta^{-J}(\hat{\mu}^J, \mu^{-J}))\rangle) \,\middle|\, \sigma^J\right] > \mathbb{E}\left[\sum_{j \in J} u^j(\langle s, \mu, \beta\rangle) \,\middle|\, \sigma^J\right]. \quad (4)$$

In this context, an arrangement, given an initial endowment e, takes the form

$$[e; \langle s, (\mu^i(\sigma^i))_{i \in I}, (\beta^i(\mu, \sigma^i))_{i \in I}\rangle; p].$$

An initial endowment e is *full revelation core-compatible* if, for every message profile $(\hat{\mu}^i(\sigma^i))_{i \in I}$ and ballot profile $(\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I}$, there exists a *full revelation core arrangement* $[e; \langle s, \mu, \beta\rangle; p]$ *associated with* e, i.e., an arrangement $[e; \langle s, \mu, \beta\rangle; p]$ such that the outcome $\langle s, \mu, \beta\rangle$ is in the full revelation core and

$$\mathbb{E}[\tilde{u}^i([e; \langle s, \mu, \beta\rangle; p])] \geq \mathbb{E}[\tilde{u}^i([e; \langle s, \hat{\mu}, \hat{\beta}\rangle; p])].$$

We now generalize the results of Theorems 1 and 2 to this more general setting.

Theorem 3. *Full revelation core outcomes are efficient.*

Proof. Suppose that $\langle s, \mu, \beta\rangle$ is not efficient. Then there exists $\bar{\beta}$ for some $\sigma \in \times_{i \in I} \Sigma^i$ such that

$$\mathbb{E}\left[\sum_{i \in I} u^i(\langle s, \mu^i, \beta\rangle) \,\middle|\, \sigma\right] < \mathbb{E}\left[\sum_{i \in I} u^i(\langle s, \mu, \bar{\beta}\rangle) \,\middle|\, \sigma\right].$$

But then taking $J = I$ and $\hat{\beta} = (\bar{\beta}, \beta_{-\sigma})$, we see that (4) is not satisfied, and hence $\langle s, \mu, \beta\rangle$ is not in the core.[9] □

Lemma 2. *The full revelation core is nonempty. In particular, every outcome $\langle s, \mu, \beta\rangle$ such that*

1. *for all $i, j \in I$, we have $s^i = a s^j$ for some $a \in \mathbb{R}_{\geq 0}$,*

[9]Note that, since $\mu = \sigma$ even under the deviation, the deviating agents can infer the full vector of signals.

2. $\mu = \sigma$*, and*

3. $\beta(\mu)$ *is efficient,*

is in the full revelation core.

Proof. Consider any such $\langle s, \mu, \beta \rangle$. It clearly satisfies the first condition of Definition 3. To see that it satisfies the second condition of Definition 3, note that, for any deviating coalition $J \subseteq I$,

$$\sum_{j \in J} \sum_{f \in F} s_f^j \mathbb{E}\left[v_f(\delta(\beta(\sigma))) \mid \sigma\right] \propto \sum_{i \in I} \sum_{f \in F} s_f^i \mathbb{E}\left[v_f(\delta(\beta(\sigma))) \mid \sigma\right].$$

Hence, maximizing the welfare of any coalition is equivalent to maximizing the welfare of the grand coalition. But, as $\beta(\mu) = \beta(\sigma)$ is efficient, the ballot profile β maximizes expected welfare. Hence, $\langle s, \mu, \beta \rangle$ is in the core. □

Theorem 4. *Every initial endowment is full revelation core-compatible.*

Proof. Consider an initial endowment vector e. Consider some message profile $(\hat{\mu}^i(\sigma^i))_{i \in I}$ and voting profile $(\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I}$ and let

$$p_f = \mathbb{E}\left[v_f(\delta((\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I}))\right],$$

that is, the expected value of the firm before any signals are revealed. Let $\langle s, \mu, \beta \rangle$ be any outcome of the form given in Lemma 2. The expected utility of agent i under the initial allocation e, message profile $(\hat{\mu}^i(\sigma^i))_{i \in I}$ and voting profile $(\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I}$ is

$$\mathbb{E}\left[\tilde{u}^i([e; \langle e, (\hat{\mu}^i(\sigma^i))_{i \in I}, (\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I} \rangle; p])\right] = \mathbb{E}\left[\sum_{f \in F} e_f^i \mathbb{E}\left[v_f(\delta((\hat{\beta}^i(\hat{\mu}(\sigma), \sigma^i))_{i \in I})) \;\middle|\; \sigma\right]\right]. \tag{5}$$

The expected utility of agent i under the arrangement $[e; \langle s, \mu, \beta \rangle; p]$ is

$$\begin{aligned}
\mathbb{E}\left[\tilde{u}^i([e; \langle s, \mu, \beta \rangle; p])\right] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{f \in F} s^i_f v_f(\delta(\beta(\mu(\sigma)))) \,\middle|\, \sigma\right] - p_f(s^i_f - e^i_f)\right] \quad (6) \\
&= \mathbb{E}\left[\sum_{f \in F} s^i_f v_f(\delta(\beta(\mu)))\right] - \mathbb{E}\left[v_f(\delta((\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I}))\right](s^i_f - e^i_f) \\
&= \sum_{f \in F} s^i_f \left(\mathbb{E}\left[v_f(\delta(\beta(\mu)))\right] - \mathbb{E}\left[v_f(\delta((\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I}))\right]\right) + \\
&\qquad \sum_{f \in F} \mathbb{E}\left[v_f(\delta((\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I}))\right] e^i_f.
\end{aligned}$$

Hence, the difference between utility under the arrangement $[e; \langle s, \mu, \beta \rangle; p]$, given by (6), and the utility under the arrangement $\tilde{u}^i([e; \langle e, (\hat{\mu}^i(\sigma^i))_{i \in I}, (\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I} \rangle; p]$, given by (5), is given by

$$\sum_{f \in F} s^i_f \left(\mathbb{E}\left[v_f(\delta(\beta(\mu)))\right] - \mathbb{E}\left[v_f(\delta((\hat{\beta}^i(\hat{\mu}, \sigma^i))_{i \in I}))\right]\right),$$

which is non-negative as $\mu = \sigma$, $\delta(\beta(\sigma))$ is efficient, and s^i is proportional to the total value of the economy for all $i \in I$. Hence,

$$\tilde{u}^i([e; \langle s, \beta \rangle; p]) \geq \tilde{u}^i([e; \langle e, \hat{\beta} \rangle; p]),$$

and so $[e; \langle s, \mu, \beta \rangle; p]$ is core-compatible. □

2 Transferrable Control Rights

We now consider the case where control rights are transferrable; that is, where r is an outcome variable. Hence, an *outcome* now takes the form $\langle s, r, \beta \rangle$, where r is a profile of control rights and, as before, s is a portfolio profile and β is a ballot profile. Since r is now endogenous, we explicitly indicate the dependence of the decision vector $\delta = \delta(r, \beta)$ on the profile of control rights r. We denote the set of all possible profiles of controls rights as

$\mathcal{R} \equiv \{r \in [0,1]^{I\times M} : \sum_{i\in I} r^i_f = 1 \text{ for all } f \in F\}$. The utility from an outcome $\langle s, r, \beta\rangle$ is now given by

$$u^i(\langle s, r, \beta\rangle) \equiv \sum_{f\in F} s^i_f v_f(\delta(r, \beta)).$$

Agents are now endowed with an initial portfolio of control rights $(k^i_m)_{i\in I, m\in M}$. We also let the column vector $(q_m)_{m\in M}$ denote the prices at which control rights for each motion trade. Hence, an arrangement now takes the form $[e, k; \langle s, r, \beta\rangle; p, q]$, where e and k are the endowed portfolio profile and profile of control rights, respectively, $\langle s, r, \beta\rangle$ is an outcome, and p and q are the prices for shares and control rights, respectively. Given an initial endowment e, k, an arrangement $[e, k; \langle s, r, \beta\rangle; p, q]$, induces a utility for agent i of

$$\tilde{u}^i([e, k; \langle s, r, \beta\rangle; p, q]) \equiv \sum_{f\in F} \left(s^i_f v_f(\delta(r, \beta)) - p_f(s^i_f - e^i_f)\right) - \sum_{m\in M} \left(q_m(r^i_m - k^i_m)\right).$$

2.1 Competitive Equilibrium

We now extend our definitions of the demand correspondence and competitive equilibrium to the setting with transferrable control rights. The demand $D^i(e, k; \beta; p, q)$ for agent i, given a set of ballots $\beta \in [0,1]^{I\times M}$, is given by

$$D^i(e, k; \beta; p, q) \equiv \underset{(s,r)\in\mathcal{S}\times\mathcal{R}}{\arg\max}\, \tilde{u}^i([e, k; \langle s, r, \beta\rangle; p, q])$$

and the demand correspondence for the entire economy, $D(e, k; \beta; p, q)$ is given by

$$D(e, k; \beta; p, q) \equiv \bigcap_{i\in I} D^i(e, k; \beta; p, q).$$

We now define the concept of *competitive equilibrium* for the setting where voting rights are tradable.

Definition 4. A *competitive equilibrium*, given an initial endowment (e, k), is an arrangement $[e, k; \langle s, r, \beta\rangle; p, q]$ such that

1. $(s, r) \in D(e, k; \beta; p, q)$,

2. β is a Nash equilibrium in undominated strategies of the induced voting game, and

3. there is no agent i and $(\hat{s}^i, \hat{r}^i, \hat{\beta}^i)$ such that

$$\tilde{u}^i([e, k; \langle(\hat{s}^i, s^{-i}), (\hat{r}^i, r^{-i}), (\hat{\beta}^i, \beta^{-i})\rangle; p, q]) > \tilde{u}^i([e, k; \langle s, r, \beta\rangle; p, q]).$$

Definition 4 is the natural generalization of Definition 1 to the case where control rights are tradable. Condition 1 states that the demand correspondence is non-empty: that is, each agent is demanding an optimal portfolio given the equilibrium voting behavior, and each agent is also demanding an optimal portfolio of control rights given his economic interests. Condition 2 states that each agent is voting optimally. Finally, Condition 3 states that no agent can strictly increase his utility by changing his share portfolio, his control rights, and his ballot.

We now show that, so long as the outcome of at least one motion affects the value of at least one firm, competitive equilibria cannot exist in settings with transferrable control rights.

Theorem 5. *Suppose that there exists at least one firm $\tilde{f} \in F$ and motion $\tilde{m} \in M$ such that the value of $\tilde{f}$ depends on the decision regarding $\tilde{m}$, i.e., such that $v_f(0, \delta_{-\tilde{m}}) \neq v_f(1, \delta_{-\tilde{m}})$ for all $\delta_{-\tilde{m}} \in \{0, 1\}^{M \setminus \{\tilde{m}\}}$. Then no competitive equilibrium exists.*

Proof. Consider any arrangement $[e, k; \langle s, r, \beta\rangle; p, q]$ and suppose it is a competitive equilibrium. We first show that for each firm $f \in F$, $p_f = v_f(\delta(r, \beta))$. There are two cases to consider:

1. If $p_f < v_f(\delta(r, \beta))$, then the first condition of Definition 4 is not satisfied—$(s, r) \notin D^i(e, k; \beta; p, q)$, as agent i is strictly better off with the portfolio $\hat{s}^i \equiv (s^i_f + 1, s^i_{-f})$ than

with s^i, as

$$\tilde{u}^i([e,k;\langle(\hat{s}^i,s^{-i}),r,\beta\rangle;p,q]) - \tilde{u}^i([e,k;\langle s,r,\beta\rangle;p,q]) = v_f(\delta(r,\beta)) - p_f > 0.$$

2. If $p_f > v_f(\delta(r,\beta))$, then the first condition of Definition 4 is not satisfied—$(s,r) \notin D^i(e,k;\beta;p,q)$, as agent i is strictly better off with the portfolio $\hat{s}^i \equiv (s^i_f - 1, s^i_{-f})$ than with s^i, as

$$\tilde{u}^i([e,k;\langle(\hat{s}^i,s^{-i}),r,\beta\rangle;p,q]) - \tilde{u}^i([e,k;\langle s,r,\beta\rangle;p,q]) = p_f - v_f(\delta(r,\beta)) > 0.$$

Hence, $p_f = v_f(\delta(r,\beta))$ for all $f \in F$. Suppose that $\delta_{\tilde{m}}(r,\beta) = 0$.[10] Hence, $p_{\tilde{f}} \neq v_{\tilde{f}}(\hat{\delta})$, where

$$\hat{\delta}_m = \begin{cases} 1 & m = \tilde{m} \\ (\delta(r,\beta))_m & \text{otherwise.} \end{cases}$$

Suppose $v_{\tilde{f}}(\hat{\delta}) > p_{\tilde{f}}$.[11] Consider a deviation by i to $(\hat{s}^i, \hat{r}^i, \hat{\beta}^i)$ where

$$\hat{r}^i_m = \begin{cases} \frac{1+\alpha_{\tilde{m}}}{2} & m = \tilde{m} \\ r^i_m & \text{otherwise,} \end{cases}$$

$$\hat{s}^i_f = \begin{cases} 1 + \frac{q_{\tilde{m}}(\hat{r}^i_{\tilde{m}} - r^i_{\tilde{m}})}{v_f(\hat{\delta}) - p_f} & f = \tilde{f} \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{\beta}^i_m = \begin{cases} 1 & m = \tilde{m} \\ \beta^i_m & \text{otherwise.} \end{cases}$$

[10]The case of $\delta_{\tilde{m}}(r,\beta) = 1$ is analogous.

[11]The case where $v_{\tilde{f}}(\hat{\delta}) < p_{\tilde{f}}$ is analogous.

Under this deviation, the utility of i is

$$\tilde{u}^i([e,k;\langle(\hat{s}^i,s^{-i}),(\hat{r}^i,r^{-i}),(\hat{\beta}^i,\beta^{-i})\rangle;p,q]) = \\ \sum_{f\in F}\Big(\hat{s}^i_f v_f(\delta((\hat{r}^i,r^{-i}),(\hat{\beta}^i,\beta^{-i}))) - p_f(\hat{s}^i_f - e^i_f)\Big) - \sum_{m\in M}\big(q_m(\hat{r}^i_m - k^i_m)\big)\,. \quad (7)$$

Noting that $\delta((\hat{r}^i,r^{-i}),(\hat{\beta}^i,\beta^{-i})) = \hat{\delta}$, the deviation utility given in (7) is given by

$$\sum_{f\in F}\Big(\hat{s}^i_f v_f(\hat{\delta}) - p_f(\hat{s}^i_f - e^i_f)\Big) - \sum_{m\in M}\big(q_m(\hat{r}^i_m - k^i_m)\big)\,. \quad (8)$$

Substituting in the values of $\hat{s}^i$ and $\hat{r}^i$ defined above, (8) becomes

$$\left(1 + \frac{q_{\tilde{m}}(\hat{r}^i_{\tilde{m}} - r^i_{\tilde{m}})}{v_{\tilde{f}}(\hat{\delta}) - p_{\tilde{f}}}\right)(v_{\tilde{f}}(\hat{\delta}) - p_{\tilde{f}}) + \sum_{f\in F} p_f e^i_f - \sum_{m\in M}\big(q_m(r^i_m - k^i_m)\big) - q_{\tilde{m}}(\hat{r}^i_{\tilde{m}} - r^i_{\tilde{m}}) \quad (9)$$

as $\hat{s}^i_f = 0$ for all $f \neq \tilde{f}$. Furthermore, noting that $p_f = v_f(\delta(r,\beta))$ from Items 1 and 2 above, we have that

$$\begin{aligned}\tilde{u}^i([e,k;\langle s,r,\beta\rangle;p,q]) &= \sum_{f\in F}\big(s^i_f v_f(\delta(r,\beta)) - p_f(s^i_f - e^i_f)\big) - \sum_{m\in M}\big(q_m(r^i_m - k^i_m)\big) \\ &= \sum_{f\in F} p_f e^i_f - \sum_{m\in M}\big(q_m(r^i_m - k^i_m)\big)\,. \quad (10)\end{aligned}$$

Combining expressions (9) and (10), we have that

$$\begin{aligned}\tilde{u}^i([e,k;\langle(\hat{s}^i,s^{-i}),(\hat{r}^i,r^{-i}),(\hat{\beta}^i,\beta^{-i})\rangle;p,q]) &= (v_{\tilde{f}}(\hat{\delta}) - p_{\tilde{f}}) + \tilde{u}^i([e,k;\langle s,r,\beta\rangle;p,q]) \\ &> \tilde{u}^i([e,k;\langle s,r,\beta\rangle;p,q])\end{aligned}$$

as $v_{\tilde{f}}(\hat{\delta}) - p_{\tilde{f}} > 0$ by assumption. Thus $[e,k;\langle(s,r,\beta\rangle;p,q]$ is not a competitive equilibrium, a contradiction. □

The key idea of the proof is that for any decision vector δ there is a unique price vector

p for shares which may support a competitive equilibrium, given by $\bar{p}_f = v_f(\delta)$: for any $p_f \neq v_f(\delta)$, any agent could buy or sell an arbitrarily large amount of stock in firm f to make an arbitrarily large profit. However, at the price vector $\bar{p}$ for shares, an agent may buy or sell an arbitrarily large amount of stock in some firm f such that $v_f(\hat{\delta}) \neq \bar{p}_f$, along with control rights sufficient to change the decision vector to $\hat{\delta}$; by doing so, that agent may make an arbitrarily large profit. Thus, no price vector can support a competitive equilibrium.

2.2 Core-Compatible Outcomes

Definition 5. An outcome $\langle s, r, \beta \rangle$ is in the *core* if there does not exist a set of agents $J \subseteq I$, and $\hat{\beta}^J$ such that

$$\sum_{j \in J} u^j(\langle s, r, (\hat{\beta}^J, \beta^{-J}) \rangle) > \sum_{j \in J} u^j(\langle s, r, \beta \rangle). \tag{11}$$

We say that an initial endowment (e, k) is *core-compatible* if, for every ballot profile $\hat{\beta}$, there exists a *core arrangement* $[e, k; \langle s, r, \beta \rangle; p, q]$ *associated with* (e, k), such that the outcome $\langle s, r, \beta \rangle$ is in the core and, for all $i \in I$,[12]

$$\tilde{u}^i([e, k; \langle s, r, \beta \rangle; p, q]) \geq \tilde{u}^i([e, k; \langle e, k, \hat{\beta} \rangle; p, q]).$$

2.2.1 Efficiency

Theorem 6. *Core outcomes are efficient.*

Proof. The proof follows as the proof of Theorem 1. □

2.2.2 Existence

We show first that the core is nonempty and characterize a subset of the core.

Lemma 3. *The core is nonempty. In particular, every outcome $\langle s, r, \beta \rangle$ such that*

[12]Note that, as in Section 1.2, we do not impose any restriction on beliefs about balloting, so long as agents' expectations about the balloting are consistent.

1. for all $i, j \in I$, we have $s^i = as^j$ for some $a \in \mathbb{R}_{\geq 0}$ and

2. β is efficient

is in the core.

Proof. Consider any such $\langle s, \beta \rangle$. To see that it satisfies the conditions of Definition 5, note that, for any deviating coalition $J \subseteq I$,

$$\sum_{j \in J} \sum_{f \in F} s^j_f v_f(\delta(r, \beta)) \propto \sum_{i \in I} \sum_{f \in F} s^i_f v_f(\delta(r, \beta)).$$

Hence, maximizing the welfare of any coalition is equivalent to maximizing the welfare of the grand coalition. But the ballot profile β maximizes welfare; hence, $\langle s, \beta \rangle$ is in the core. □

Now, we show that, for any initial endowment, the core may be reached through mutually beneficial exchange.

Theorem 7. *Every initial endowment is core-compatible.*

Proof. Consider an initial endowment (e, k). Consider an efficient decision vector δ and let $\beta^i_m = \delta_m$. Consider some $\hat{\beta}$ and let $p_f = v_f(\delta(k, \hat{\beta}))$ and $q_f = 0$. Let s be any portfolio profile such that $s^i_f = s^i_{f'} > 0$ for all $i \in I$ and $f, f' \in F$. By Lemma 3, $\langle s, k, \beta \rangle$ is in the core.

The utility of agent i under the initial allocation and voting profile $\hat{\beta}$ is

$$\tilde{u}^i([e, k; \langle e, k, \hat{\beta} \rangle; p, q]) = \sum_{f \in F} e^i_f v_f(\delta(k, \hat{\beta})). \tag{12}$$

The utility of agent i under the arrangement $[e, k; \langle s, k, \beta\rangle; p, q]$ is

$$\begin{aligned}
\tilde{u}^i([e, k; \langle s, k, \beta\rangle; p, q]) &= \sum_{f\in F} \left(s^i_f v_f(\delta(k, \beta)) - p_f(s^i_f - e^i_f)\right) \\
&= \sum_{f\in F} \left(s^i_f v_f(\delta(k, \beta)) - v_f(\delta(k, \hat{\beta}))(s^i_f - e^i_f)\right) \\
&= \sum_{f\in F} s^i_f(v_f(\delta(k, \beta)) - v_f(\delta(k, \hat{\beta}))) + \sum_{f\in F} v_f(\delta(k, \hat{\beta}))e^i_f \qquad (13)
\end{aligned}$$

Hence, the difference between utility under the arrangement $[e, k; \langle s, k, \beta\rangle; p, q]$, given by (13), and the utility under the arrangement $[e, k; \langle e, k, \hat{\beta}\rangle; p, q]$, given by (12), is given by

$$\sum_{f\in F} s^i_f(v_f(\delta(k, \beta)) - v_f(\delta(k, \hat{\beta})))$$

which is non-negative as $\delta(k, \beta)$ is efficient and $s^i_f = s^i_{f'} > 0$ for all $i \in I$ and $f, f' \in F$. Hence,

$$\tilde{u}^i([e, k; \langle s, k, \beta\rangle; p, q]) \geq \tilde{u}^i([e, k; \langle e, k, \hat{\beta}\rangle; p, q]).$$

and so (e, k) is core-compatible. □

The intution behind the existence of a core-compatible arrangement for any starting endowment is exactly the same as that presented in Section 1.2.2 for the case without transferrable control rights. To reach a core outcome in this setting, agents may trade both stock and control rights, taking into account how such trades may affect voting outcomes. After such trades, in any core outcome, the voting incentives of all agents will be roughly aligned with market efficiency; that is, for any set of agents with majority control, their welfare will be maximized by the efficiency-maximizing decision.

We now return to the setting of Example 1, only now we allow for tradable control rights.

In this setting, the initial endowment is given by

$$e = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and initial control rights are (as in Example 1) given by $k = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)^{\top}$. One core-compatible arrangement for this setting is then given by

$$\left[e, k, \left\langle \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right), (1,1,1,1) \right\rangle, \begin{pmatrix} 16 \\ 6 \end{pmatrix}, (0) \right].$$

Note that in this core-compatible arrangement, control rights are traded in the associated outcome, but are priced at zero. Control rights are priced at 0 in this example as all agents vote identically.

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