Multilateral Matching: Corrigendum *

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Abstract

We identify an error in the claim by Hatfield and Kominers (2015) that every stable outcome is efficient in the setting of multilateral matching with contracts. We then show that the result can be recovered under a suitable differentiability condition.

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1 Introduction

Hatfield and Kominers (2015) introduced a model of matching in networks with continuously divisible multilateral contracts and transferrable utility. They showed that competitive equilibria exist and are efficient when agents' valuations are concave and also claimed that (under the same condition) competitive equilibria correspond to stable outcomes. However, as we show here, the correspondence of stability and competitive equilibrium in the multilateral matching framework requires an additional differentiability condition.

2 Model

There is a finite set I of *agents*, and a finite set Ω of *ventures*. Each venture $\omega \in \Omega$ is associated with a set of at least two agents $a(\omega) \subseteq I$; there may be several ventures associated with the same set of agents. For a set of ventures $\Psi \subseteq \Omega$, we denote by $a(\Psi) \equiv \cup_{\psi \in \Psi} a(\psi)$ the set of agents associated with ventures in Ψ. We denote by $\Psi_i \equiv \{\psi \in \Psi : i \in a(\psi)\}$ the set of ventures in Ψ associated with agent i. We denote by $r_\omega \in [0, r_\omega^{\max}]$ the chosen level of *participation* in venture $\omega \in \Omega$ by the agents in $a(\omega)$.

Each agent $i \in I$ has a continuous valuation function $v^i(r)$ over ventures, where the vector $r \equiv (r_\omega)_{\omega \in \Omega}$ is an *allocation*, which indicates the investment in each venture $\omega \in \Omega$. We assume that v^i is unaffected by ventures to which i is not a party, i.e., $v^i(r_\omega, r_{\Omega \smallsetminus \{\omega\}}) = v^i(\tilde{r}_\omega, r_{\Omega \smallsetminus \{\omega\}})$ for all ω such that $i \notin a(\omega)$. An allocation $\hat{r}$ is *efficient* if it maximizes total surplus, i.e.,

$$\hat{r} \in \underset{0 \le r \le r^{\max}}{\arg\max} \left\{ \sum_{i \in I} v^i(r) \right\}.$$

We denote by $p \equiv (p_\omega^i)_{i \in I, \omega \in \Omega}$ the *price* matrix, for which p_ω^i is the per-unit transfer from agent i when he engages in venture ω. For any agent $j \notin a(\omega)$, we use the convention that $p_\omega^j \equiv 0$. Furthermore, a venture does not create or use the numeraire; hence $\sum_{i \in I} p_\omega^i = 0$ for all $\omega \in \Omega$.

An allocation r along with a price matrix p together define an *arrangement* $[r;p]$. The *utility function* $u^i([r;p])$ of an agent i is quasilinear over ventures and transfer prices; hence, it can be expressed in the form

$$u^i([r;p]) \equiv v^i(r) - p^i \cdot r.$$

Given prices p, we define the *demand correspondence* $D^i(p)$ for agent i as

$$D^i(p) \equiv \underset{0 \le r \le r^{\max}}{\arg\max} \{u^i([r;p])\}$$

and the demand correspondence for the entire economy as

$$D(p) \equiv \bigcap_{i \in I} D^i(p).$$

An arrangement $[r;p]$ is a *competitive equilibrium* if $r \in D(p)$.

A *contract* x is comprised of a venture $\omega \in \Omega$, a size of that venture $r_\omega \in [0, r_\omega^{\max}]$, and a transfer vector $s_\omega \in \mathbb{R}^I$ (where we set $s_\omega^j = 0$ for all $j \notin a(\omega)$). The set of all contracts is

$$X \equiv \left\{ (\omega, r_\omega, s_\omega) \in \Omega \times \mathbb{R}_{\ge 0} \times \mathbb{R}^I : r_\omega \le r_\omega^{\max},\ s_\omega^i = 0 \text{ for } i \notin a(\omega), \text{ and } \sum_{i \in I} s_\omega^i = 0 \right\}.$$

For $x = (\omega, r_\omega, s_\omega) \in X$, we let $\tau(x) \equiv \omega$ denote the venture associated with x; for $Y \subseteq X$ we let $\tau(Y) \equiv \cup_{y \in Y}\{\tau(y)\}$. For a contract $x \in X$ we let $a(x) \equiv a(\tau(x))$ and for $Y \subseteq X$ we let $a(Y) \equiv a(\tau(Y))$; we let $Y_i \equiv \{y \in Y : i \in a(y)\}$.

A set of contracts $Y \subseteq X$ is an *outcome* if it describes a well-defined participation and pricing plan, i.e., if for any distinct $(\omega, r_\omega, s_\omega), (\bar{\omega}, \bar{r}_{\bar{\omega}}, \bar{s}_{\bar{\omega}}) \in Y$, we have that $\omega \neq \bar{\omega}$.

For a given outcome Y, we define $\rho(Y)$ as

$$\rho_\omega(Y) \equiv \begin{cases} r_\omega & (\omega, r_\omega, s_\omega) \in Y \\ 0 & \text{otherwise;} \end{cases}$$

that is, $\rho(Y)$ denotes the associated allocation vector of venture participations. Similarly, we define $\pi(Y)$ as

$$\pi^j_\omega(Y) \equiv \begin{cases} \frac{s^j_\omega}{r_\omega} & (\omega, r_\omega, s_\omega) \in Y \\ 0 & \text{otherwise,} \end{cases}$$

denote the matrix of per-unit transfer prices associated to Y. The *utility* from an outcome Y for agent i is then given by

$$u^i(Y) \equiv v^i(\rho(Y)) - \pi^i(Y) \cdot \rho(Y).$$

The *choice correspondence* of agent i is given by

$$C^i(Y) \equiv \underset{Z \subseteq Y_i; Z \text{ is an outcome}}{\arg\max} \{u^i(Z)\}.$$

An allocation is *stable* if it is

1. *individually rational*: for all $i \in I$, $A_i \in C^i(A)$.

2. *unblocked*: there does not exist a nonempty $Z \subseteq X \smallsetminus A$ such that for all $i \in a(Z)$ we have that $Z_i \subseteq Y^i$ for all $Y^i \in C^i(Z \cup A)$.

3 Counterexample

Hatfield and Kominers (2015) claimed the following result.

Theorem 8 (Original). Suppose that agents' valuation functions are concave. Then, for any stable outcome A, the allocation $\rho(A)$ is efficient.

Here, we show by way of example that the result is not correct as stated.

Example 1. Let $I = \{i, j\}$, $\Omega = \{\psi, \omega\}$, $r^{\max}_{\psi} = r^{\max}_{\omega} = 1$, and $a(\psi) = a(\omega) = \{i, j\}$. Let

$$v^i(r) = 3 \min\{r_\psi, r_\omega\}$$
$$v^j(r) = \min\{0, 1 - (r_\psi + r_\omega)\};$$

note that both v^i and v^j are concave.

Let $x = (\psi, \frac{1}{2}, (0, 0))$ and $y = (\omega, \frac{1}{2}, (0, 0))$, and let $A = \{x, y\}$. We have that $C^i(A) = \{A\}$ while $C^j(A) = \{A, \{x\}, \{y\}, \varnothing\}$ and so A is individually rational.

Now, suppose that $Z \subseteq X \smallsetminus A$ blocks A. Then there exist $Y^i \in C^i(A \cup Z)$ and $Y^j \in C^j(A \cup Z)$ such that $u^i(Y^i) + u^j(Y^j) > \frac{3}{2}$ (as $u^i(A) + u^j(A) = \frac{3}{2}$). Since $v^j(r) \leq 0$ for all r, we have that $v^i(\rho(Y^i)) > \frac{3}{2}$. Hence, $\rho_\psi(Y^i) > \frac{1}{2}$ and $\rho_\omega(Y^i) > \frac{1}{2}$; these facts imply that, since $\rho_\psi(A) = \rho_\omega(A) = \frac{1}{2}$, we have that $\tau(Z) = \{\psi, \omega\}$, i.e., Z has a contract specifying venture ψ and a contract specifying venture ω. Thus, $\{Z\} = C^i(A \cup Z)$ and $\{Z\} = C^j(A \cup Z)$ as i and j must choose at most one contract associated with each venture; let $Z = \{(\psi, \bar{r}_\psi, (t_\psi, -t_\psi)), (\omega, \bar{r}_\omega, (t_\omega, -t_\omega))\}$ (where we use the convention that transfers are listed in alphabetical order).

Let $\bar{Z} = \{(\psi, \bar{r}^{\min}, (t_\psi, -t_\psi)), (\omega, \bar{r}^{\min}, (t_\omega, -t_\omega))\}$ where $\bar{r}^{\min} = \min\{\bar{r}_\psi, \bar{r}_\omega\} > \frac{1}{2}$; it is immediate that $\bar{Z}$ is a blocking set (given that Z is) and so $\{\bar{Z}\} = C^i(A \cup \bar{Z})$ and $\{\bar{Z}\} = C^j(A \cup \bar{Z})$.

For j to prefer $\bar{Z}$ to $\{(\psi, \bar{r}^{\min}, (t_\psi, -t_\psi))\}$, we require that

$$1 - (\bar{r}^{\min} + \bar{r}^{\min}) + t_\psi + t_\omega > t_\psi \tag{1}$$
$$t_\omega \geq 2\bar{r}^{\min} - 1.$$

Similarly, for j to prefer $\bar{Z}$ to $\{(\omega, \bar{r}^{\min}, (t_\omega, -t_\omega))\}$, we require that

$$1 - (\bar{r}^{\min} + \bar{r}^{\min}) + t_\psi + t_\omega > t_\omega \tag{2}$$
$$t_\psi \geq 2\bar{r}^{\min} - 1.$$

Finally, for i to prefer $\bar{Z}$ to A, we require that

$$3\bar{r}^{\min} - t_\psi - t_\omega > \frac{3}{2}. \tag{3}$$

Since $\bar{r}^{\min} > \frac{1}{2}$, it is immediate that (1)–(3) can not hold simultaneously.

Thus, we conclude that A must be unblocked, and hence stable; yet A is not efficient, as $\rho(A) = (\frac{1}{2}, \frac{1}{2})$, which yields total welfare of $\frac{3}{2}$, while $r = (1, 1)$ yields total welfare of 2.

As all competitive equilibria in our setting are efficient (Theorem 1 of Hatfield and Kominers (2015)), and the outcome A of Example 1 is not efficient, it is immediate that the following corollary of Theorem 8 of Hatfield and Kominers (2015) is also incorrect as stated.

Corollary 2 (Original). Suppose that agents' valuation functions are concave. Then, for any stable outcome A, there exists a price matrix p such that the arrangement $[\rho(A); p]$ is a competitive equilibrium.

4 Corrected Result

The proof of Theorem 8 presented by Hatfield and Kominers (2015) required that, at any inefficient allocation r, there exists at least one venture ψ such that a slight increase or decrease in the level of participation in ψ increases welfare. This fails in Example 1: Even though $r = (\frac{1}{2}, \frac{1}{2})$ is not efficient, welfare decreases if increase or decrease participation in just ψ or just ω; only simultaneously increasing both ψ and ω increases welfare. However, when each valuation function is differentiable, then welfare is also differentiable; hence, at any allocation that is not a local maximum, there *does* exist at least one venture ψ such that a slight increase or decrease in the level of participation in ψ increases welfare. Thus, when agents' valuation functions are concave and differentiable, we recover the conclusion of Theorem 8.

Specifically, we say that a valuation function is *differentiable* if, at every point $r \in \times_{\omega\in\Omega}[0, r_\omega^{\max}]$, there exists a vector $q \in \mathbb{R}^\Omega$ such that, for every sequence $r^1, r^2, \ldots$ that approaches r (and where each $r^n \in \times_{\omega\in\Omega}[0, r_\omega^{\max}]$) we have that

$$\lim_{n\to\infty} \frac{|(v(r^n) - v(r)) - q \cdot (r^n - r)|}{\|r^n - r\|} = 0.$$

Theorem 8 (Corrected). Suppose that agents' valuation functions are concave *and differentiable*. Then, for any stable outcome A, the allocation $\rho(A)$ is efficient.

With the additional assumption of differentiability, Corollary 2 then follows from Theorem 8 as before.

Corollary 2 (Corrected). Suppose that agents' valuation functions are concave *and differentiable*. Then, for any stable outcome A, there exists a price matrix p such that the arrangement $[\rho(A); p]$ is a competitive equilibrium.

References

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